4. An Application of the Perturbation Theorem for m-Accretive Operators. II

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1. Introduction and statement of the result. This note is concerned with the homogeneous Dirichlet problem for a nonlinear elliptic equation

$$(1)\qquad -\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\right)+\beta(x,u)=f \qquad \text{on } \Omega,$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary.

Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space. We consider only real-valued functions in the case of $p\geqq 2$. Then it follows from the Poincaré inequality that $W_0^{1,p}(\Omega)\subset L^2(\Omega)$. Setting

$$\phi(u)=\frac{1}{p}\sum_{j=1}^{n}\int_{\Omega}\left|\frac{\partial u}{\partial x_j}\right|^p dx \qquad \text{for } u\in W_0^{1,p}(\Omega)$$

and $\phi(u)=+\infty$ otherwise, ϕ is a proper lower semicontinuous convex function on $L^2(\Omega)$. The subdifferential $\partial\phi$ of ϕ is given by

$$\partial\phi(u)=-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\right) \qquad \text{for } u\in D(\partial\phi)\subset W_0^{1,p}(\Omega)$$

and is m-accretive in $L^2(\Omega)$ (see e.g. [1] or [2]).

Let $\beta\in C^1(\Omega\times J)$, where J is an open interval on $\boldsymbol{R}$ containing the origin. We assume that

(i) $\beta(x,0)=0$ for every $x\in\Omega$, and $\partial\beta/\partial s\geqq 0$ on $\Omega\times J$.

(ii) for every $x\in\Omega$, $\beta(x,\cdot):J\to\boldsymbol{R}$ is onto.

Then we can introduce the m-accretive operator $\tilde{\beta}$ in $L^2(\Omega)$:

$$D(\tilde{\beta})=\{u\in L^2(\Omega)\,;\,u(x)\in J \text{ (a.e. on } \Omega),\ \beta(x,u(x))\in L^2(\Omega)\},$$
$$\tilde{\beta}u(x)=\beta(x,u(x)) \qquad \text{for } u\in D(\tilde{\beta}).$$

The purpose of this note is to prove the following

Theorem 1. *Let $A=\partial\phi$ and $B=\tilde{\beta}$ be m-accretive operators as above. Assume that there are nonnegative constants c, a and b $[b<p^p(p-1)^{-(p-1)}]$ such that on $\Omega\times J$*

$$(2)\qquad \sum_{j=1}^{n}\left|\frac{\partial\beta}{\partial x_j}(x,s)\right|^p\leqq\{c+as^2+b[\beta(x,s)]^2\}\left[\frac{\partial\beta}{\partial s}(x,s)\right]^{p-1}.$$

Then $A+B=\partial\phi+\tilde{\beta}$ with domain $D(A)\cap D(B)$ is m-accretive in $L^2(\Omega)$.

Noting that $A=\partial\phi$ is strictly accretive (for a precise estimate see Simon [7]), we obtain

Corollary 2. *For every $f\in L^2(\Omega)$ there exists a unique solution $u\in D(\partial\phi)\cap D(\tilde{\beta})$ of the equation* (1).

Remark 3. Let

$$D(\psi)=\left\{u \in L^2(\Omega)\,;\, u(x) \in J \text{ (a.e. on } \Omega),\ \int_0^{u(x)} \beta(x, s)ds \in L^1(\Omega)\right\}.$$

Setting

$$\psi(u)=\int_\Omega \int_0^{u(x)} \beta(x, s)dsdx \qquad \text{for } u \in D(\psi)$$

and $\psi(u)=+\infty$ otherwise, we see that $\tilde{\beta}$ is the subdifferential of $\psi: \tilde{\beta}=\partial\psi$. Therefore, Theorem 1 implies that

$$\partial(\phi+\psi)=\partial\phi+\partial\psi.$$

2. Proofs. We first note that $\tilde{\beta}$ is m-accretive in $L^2(\Omega)$ if conditions (i) and (ii) are satisfied. In fact, let $v \in L^2(\Omega)$. Then for almost all $x \in \Omega$ the equation

$$s+\beta(x, s)=v(x)$$

has a unique solution $s=u(x)$ such that $|u(x)|\leqq|v(x)|$. Therefore, $u \in D(\tilde{\beta})$ and $v(x)=(1+\tilde{\beta})u(x)$.

Let $u \in C_0^1(\bar{\Omega})$ and $\varepsilon>0$. Setting $w(x)=(1+\varepsilon\tilde{\beta})^{-1}u(x)$, we see from the implicit function theorem that $w \in C_0^1(\bar{\Omega})$ and

$$(3) \qquad \frac{\partial w}{\partial x_j}(x)=\left[1+\varepsilon\frac{\partial\beta}{\partial s}(x, w(x))\right]^{-1}\left[\frac{\partial u}{\partial x_j}(x)-\varepsilon\frac{\partial\beta}{\partial x_j}(x, w(x))\right].$$

So, we have

$$\left|\frac{\partial w}{\partial x_j}\right|^p\leqq 2^{p-1}\left[\left|\frac{\partial u}{\partial x_j}\right|^p+\varepsilon\left(\frac{\partial\beta}{\partial s}\right)^{-(p-1)}\left|\frac{\partial\beta}{\partial x_j}\right|^p\right]$$

and hence

$$(4) \qquad \phi(w)\leqq 2^{p-1}\phi(u)+2^{p-1}\frac{\varepsilon}{p}\int_\Omega\left(\frac{\partial\beta}{\partial s}\right)^{-(p-1)}\sum_{j=1}^n\left|\frac{\partial\beta}{\partial x_j}\right|^p dx.$$

Now let $B=\tilde{\beta}$. Then we have

Lemma 4. *$W_0^{1,p}(\Omega)$ is invariant under $(1+\varepsilon B)^{-1}$, $\varepsilon>0$, if the assumption of Theorem 1 is satisfied.*

Proof. We may assume that $\partial\beta/\partial s\geqq 1$ on $\Omega\times J$. In fact, $\beta(x, s)$ in (2) can be replaced by $\beta(x, s)+s$. We see from (4) and (2) that for $u \in C_0^1(\bar{\Omega})$

$$\phi((1+\varepsilon B)^{-1}u)\leqq 2^{p-1}\phi(u)+2^{p-1}p^{-1}\varepsilon[c\mu(\Omega)+a\,\|u\|^2+b\,\|B_\varepsilon u\|^2],$$

where

$$\mu(\Omega)=\int_\Omega dx$$

and B_ε is the Yosida approximation of B:

$$B_\varepsilon u(x)=\varepsilon^{-1}[u(x)-(1+\varepsilon B)^{-1}u(x)]=\beta(x, w(x))\,;$$

note further that $\|w\|\leqq\|u\|$.

Let $u \in W_0^{1,p}(\Omega)$. Then there is a sequence $\{u_m\}\subset C_0^1(\bar{\Omega})$ such that $u_m\to u$ $(m\to\infty)$ in $W_0^{1,p}(\Omega)$. Noting that

$$(1+\varepsilon B)^{-1}u_m\to(1+\varepsilon B)^{-1}u \ (m\to\infty) \qquad \text{in } L^2(\Omega),$$

we see from the lower semicontinuity of ϕ that

$$\phi((1+\varepsilon B)^{-1}u)\leqq\liminf_{m\to\infty}\phi((1+\varepsilon B)^{-1}u_m)$$
$$\leqq 2^{p-1}\phi(u)+2^{p-1}p^{-1}\varepsilon[c\mu(\Omega)+a\|u\|^2+b\|B_\varepsilon u\|^2],$$

i.e., $(1+\varepsilon B)^{-1}u\in W_0^{1,p}(\Omega)$. Q.E.D.

The proof of Theorem 1 is based on the following

Lemma 5 (cf. [5]). *Let A and B be m-accretive operators in $L^2(\Omega)$, with $D(A)\cap D(B)$ non-empty. Assume that there exist a constant b $(0\leqq b<1)$ and a nondecreasing function $\psi_0(r)\geqq 0$ of $r\geqq 0$ such that for all $u\in D(A)$ and $\varepsilon>0$,*

$$(Au, B_\varepsilon u)\geqq -\psi_0(\|u\|)-b\|B_\varepsilon u\|^2.$$

Then $A+B$ is also m-accretive in $L^2(\Omega)$.

This lemma holds even if B is multi-valued.

Proof of Theorem 1. Let $A=\partial\phi$ and $B=\tilde{\beta}$. We shall show that for all $u\in D(A)$ and $\varepsilon>0$,

(5) $\quad (Au, B_\varepsilon u)\geqq -p^{-p}(p-1)^{p-1}[c\mu(\Omega)+a\|u\|^2+b\|B_\varepsilon u\|^2].$

Let $u\in D(A)$. Then $u\in W_0^{1,p}(\Omega)$. Setting $w(x)=(1+\varepsilon B)^{-1}u(x)$, we see from Lemma 4 that $w\in W_0^{1,p}(\Omega)$ and hence (3) holds for almost all $x\in\Omega$. So, we have

$$(Au, B_\varepsilon u)=-\int_\Omega\sum_{j=1}^n\frac{\partial}{\partial x_j}\left(\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\right)\varepsilon^{-1}[u(x)-w(x)]dx$$
$$=\sum_{j=1}^n\int_\Omega\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\left(1+\varepsilon\frac{\partial\beta}{\partial s}\right)^{-1}\left(\frac{\partial\beta}{\partial x_j}+\frac{\partial\beta}{\partial s}\frac{\partial u}{\partial x_j}\right)dx$$
$$\geqq\int_\Omega\left(1+\varepsilon\frac{\partial\beta}{\partial s}\right)^{-1}\frac{\partial\beta}{\partial s}\sum_{j=1}^n\left|\frac{\partial u}{\partial x_j}\right|^p dx$$
$$-\int_\Omega\left(1+\varepsilon\frac{\partial\beta}{\partial s}\right)^{-1}\sum_{j=1}^n\left|\frac{\partial u}{\partial x_j}\right|^{p-1}\left|\frac{\partial\beta}{\partial x_j}\right|dx.$$

Therefore, we obtain

$$(Au, B_\varepsilon u)\geqq-\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}\int_\Omega\left(\frac{\partial\beta}{\partial s}\right)^{-(p-1)}\sum_{j=1}^n\left|\frac{\partial\beta}{\partial x_j}\right|^p dx,$$

where we have assumed that $\partial\beta/\partial s\geqq 1$ on $\Omega\times J$. Consequently, (5) follows from (2). Q.E.D.

3. Remarks. (I) If in particular $J=\boldsymbol{R}$, then condition (ii) imposed on β is unnecessary.

(II) In Theorem 1 suppose that $p=2$ and $c=0$ in (1). Then the assertion is true even if $\Omega=\boldsymbol{R}^n$ (see Okazawa [6]). In this case we see that $H^1(\boldsymbol{R}^n)$ is invariant under $(1+\varepsilon B)^{-1}$, $\varepsilon>0$.

(III) Let γ be a multi-valued m-accretive operator in $\boldsymbol{R}$; namely, γ be a maximal monotone set in $\boldsymbol{R}\times\boldsymbol{R}$. Assume that $0\in D(\gamma)$ and $0\in\gamma(0)$. Let $\tilde{\gamma}$ be the associated m-accretive operator in $L^2(\Omega)$:

$$D(\tilde{\gamma})=\{u\in L^2(\Omega);\text{ there is } v\in L^2(\Omega)\text{ such that } v(x)\in\gamma(u(x))\text{ a.e. on }\Omega\},$$
$$\tilde{\gamma}u(x)=\gamma(u(x))\qquad\text{for } u\in D(\tilde{\gamma}).$$

Then we have

Theorem 6. *Let $A+B$ be the m-accretive operator obtained in Theorem* 1, *and $C=\tilde{\gamma}$. Then $A+B+C=\partial\phi+\tilde{\beta}+\tilde{\gamma}$ is also m-accretive in $L^2(\Omega)$.*

In fact, we have

$$((A+B)u, C_\varepsilon u)\geqq(Au, C_\varepsilon u)$$
$$=\int_\Omega \gamma'_\varepsilon(u(x))\sum_{j=1}^n\left|\frac{\partial u}{\partial x_j}\right|^p dx\geqq 0.$$

We note that Theorem 6 is a generalization of Theorem 3.1 in Brezis-Crandall-Pazy [3]. For another generalization we refer to Konishi [4] and Barbu [1].

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