# 27. On the T-Genus of Knot Cobordism

By Akio KAWAUCHI, Hitoshi MURAKAMI, and Kouji SUGISHITA
Department of Mathematics, Osaka City University

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The second and third authors introduced in [4] an integral invariant $T(k)$ of a classical (tame) knot $k$ such that (1) $T(k)$ is invariant under knot cobordism, (2) $g^*(k) \leq T(k)$ and (3) $T(k) \equiv \mathrm{Arf}\,(k) \pmod 2$, where $g^*(k)$ and Arf $(k)$ are the slice genus and the Arf invariant of $k$, respectively. We call $T(k)$ the *T-genus* of $k$. The purpose of this paper is to give an alternative definition of the $T$-genus and to note that the $T$-genus induces a metric function $d_T$ on the knot cobordism group $X$ defined by Fox-Milnor in [2]. Some properties of the space $(X, d_T)$ are described without proof here, but more properties containing the details will appear in "On a geometry of the knot cobordism group".

Let $R$ be the Borromean rings (cf. Fox [1, p. 131]). We denote by $k\sharp_b(R_1+\cdots+R_r)$ a knot obtained by a fusion from the split union $k+R_1+\cdots+R_r$ of a knot $k$ and $r$ copies $R_1, \cdots, R_r$ of $R$ (see [3] for the definition of fusion). Note that the knot type of the resulting knot depends on a choice of the fusion-bands.

**Lemma 1.** *Given a knot $k$ with $T(k) \geq 1$, there is a knot $k' = k\sharp_b R$ such that $T(k') \leq T(k)-1$.*

*Proof.* Let $T(k)=r$. By [4, Proof of Theorem (2)] there is a cobordism surface of genus 0 between $k$ and $R_1+\cdots+R_r$. Then we obtain a cobordism surface of genus 0 between $k+R_1$ and $R_2+\cdots+R_r$. By the deformation theory [3] of cobordism surface, some $k'=k\sharp_b R_1$ is cobordant to some $k''=0\sharp_b(R_2+\cdots+R_r)$ (0 is the trivial knot). Since $T(k')=T(k'')$ and $T(k'') \leq r-1$, the desired result follows.

For a knot $k$ the minimal number of $r$ such that some $k\sharp_b(R_1+\cdots+R_r)$ is a slice knot is denoted by $B(k)$.

**Theorem 2.** $T(k)=B(k)$.

*Proof.* By Lemma 1 $T(k) \geq B(k)$, since $(\cdots((k\sharp_b R_1)\sharp_b R_2)\cdots)\sharp_b R_r$ is modified as $k\sharp_b(R_1+\cdots+R_r)$ by deforming and sliding the fusion-bands (cf. [3, Lemma 1.14]). To see that $T(k) \leq B(k)$, let $B(k)=s$. Since some $k\sharp_b(R_1+\cdots+R_s)$ is slice, $k+R_1+\cdots+R_s$ bounds a surface of genus 0 in $R^3[0, +\infty)$. So there is a cobordism surface of genus 0 between $k$ and $R_1+\cdots+R_s$. By the deformation theory [3], $k$ is cobordant to some $k'=0\sharp_b(R_1+\cdots+R_s)$. Then $T(k)=T(k') \leq s=B(k)$, completing the proof.

For an element $x=[k]$ of the knot cobordism group $X$, we let $T(x)=T(k)$. Define a function $d_T: X\times X\to\{0, 1, 2, 3, \cdots\}$ by $d_T(x, y)=T(x-y)$ for all $x, y$ in $X$.

**Theorem 3.** *The function $d_T$ is a metric function on $X$.*

*Proof.* From [4] or Theorem 2 we see that (1) $T(x)\geq 0$ ($^\forall x\in X$) and $T(x)=0$ iff $x=0$, (2) $T(-x)=T(x)$ ($^\forall x\in X$) and (3) $T(x+y)\leq T(x)+T(y)$ ($^\forall x, {}^\forall y\in X$). Then it is easily checked that (1)′ $d_T(x, y)\geq 0$ ($^\forall x, {}^\forall y\in X$) and $d_T(x, y)=0$ iff $x=y$, (2)′ $d_T(x, y)=d_T(y, x)$ ($^\forall x, {}^\forall y\in X$) and (3)′ $d_T(x, y)+d_T(y, z)\geq d_T(x, z)$ ($^\forall x, {}^\forall y, {}^\forall z\in X$). This completes the proof.

**Corollary 4.** $|T(x)-T(y)|\leq T(x+y)\leq T(x)+T(y)$ *for all* $x, y\in X$.

For any claim stated below, no proof will be given here.

**Claim 5.** *For any* $x=[k]$ *and* $x'=[k\#_b R]$, $d_T(x, x')=|T(x)-T(x')|=1$.

By Lemma 1 and Claim 5, when $T(x)\geq 1$, we have an $x'$ such that $T(x')=T(x)-1$. *For any $x$ can one always find an $x'$ such that* $T(x')=T(x)+1$? (The answer is yes if $T(x)\leq 1$.) Let $S(x)$ be the unit sphere, $\{y\in X\,|\,d_T(x, y)=1\}$ around $x$.

**Claim 6.** (1) $S(x)=x+S(0)=\{x+y\,|\,y\in S(0)\}$, (2) $S(x)$ *is an infinite set*, (3) $X=\bigcup_{x\in X} S(x)$ *and* (4) $S(x)\cap S(y)\neq\phi$ *iff* $x=y$ *or* $d_T(x, y)=2$.

*Is there a pair $x, y$ with $d_T(x, y)=2$ such that $S(x)=S(y)$? For any pair $x, y$ with $d_T(x, y)=2$, does $S(x)\cap S(y)$ contain at least two points? Is it an infinite set?* Let $k_n$ be the double knot with $n$ full twists, so that $k_{-1}$, $k_0$, $k_1$ and $k_2$ are the trefoil, trivial, figure eight and stevedore knots, respectively. Let $a_n=[k_{n+1}]$. Noting the index, $a_{\pm 1}=0$.

**Claim 7.** $T(a_n)\leq||n|-1|$, $d_T(a_n, a_{n-1})=1$ *and for* $n\neq 0$, $d_T(a_{n-1}, a_{n+1})=2$.

It is conjectured that *the above inequality is replaced by the equality*, whereas $g^*(k_{n+1})\leq 1$. It is true when $|n|\leq 3$. A sequence $(x_0, x_1, \cdots, x_n)$ of points $x_i$ in $X$ with $x_i\neq x_{i+1}$ for all $i$ is called a *polygon*. The *curvature* of a polygon $L=(x_0, x_1, \cdots, x_n)$ at $x_i$, $i\neq 0, n$, denoted by $\theta^i=\theta(L, x_i)$ is defined by

$$\cos(\pi-\theta^i)=-\cos\theta^i=\frac{d_T(x_{i-1}, x_i)^2+d_T(x_i, x_{i+1})^2-d_T(x_{i-1}, x_{i+1})^2}{2d_T(x_{i-1}, x_i)d_T(x_i, x_{i+1})}$$

and $0\leq\theta^i\leq\pi$. The sum $\theta=\theta(L)=\sum_{i=1}^{n-1}\theta^i$ is called the *total curvature* of $L$.

**Claim 8.** (1) *For any* $y, z\in S(x)$ *with* $y\neq z$, *the curvature* $\theta((y, x, z), x)=0$, (2) *for any $x, y$ in $X$ with* $d_T(x, y)=d\geq 2$, *there exists a polygon* $(x_0, x_1, \cdots, x_d)$ *of total curvature* 0 *such that* $x_0=x$, $x_d=y$ *and* $d_T(x_i, x_j)=|i-j|$ *for all* $i, j$. *Moreover, if* $y-x\neq dz$ *for any* $z\in S(0)$, *then at least two such polygons exist.*

The curvature of the polygon $(x, 0, -x)$ at 0 is also called the *refraction* of $x$. For example, let $b_n = na_{-2}$ and $c_n = b_n + a_0$ for $n \geq 1$. We have that $T(2b_n) = 2T(b_n) = 2n$, $T(2c_n) = 2n$ and $T(c_n) = n+1$. The refraction of $b_n$ is 0, but the refraction $\theta_n$ of $c_n$ is given by the identity $\cos\theta_n = (n^2 - 2n - 1)/(n+1)^2$. Finally, J. Tao pointed out that the space $(X, d_T)$ is regarded as a tolerance space defined by Zeeman in [5], where the tolerance relation $\sim$ is given by the following: $x \sim y$ if and only if $d_T(x, y) \leq 1$.

## References

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