25. *Iterated Log Type Strong Limit Theorems for Self-Similar Processes*

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1. Let $\{X(t,\omega)\,;\,0\leq t<+\infty\}$ be a real valued separable, measurable, stochastically continuous self-similar process of order $H>0$, where "self-similar process" means that for any $a>0$, $\{X(t)\}$ and $\{a^{-H}X(at)\}$ have the same finite dimensional distribution. We will denote it by $X(t)\overset{d}{=}a^{-H}X(at)$. Set

$$Y(\omega)=\sup_{0\leq t\leq 1}|X(t,\omega)|.$$

Theorem 1. *Let $f(x)$ be a positive, continuous, non-decreasing function defined on the positive half line. Assume that $E[f(Y)]$ is finite. Let $\phi(x)$ be a positive, continuous function defined on the positive half line which satisfies the following conditions;*

(i) *$\phi(x)$ is non-decreasing,*

(ii) $\lim_{x\downarrow 1}\sup_{n=1,2,\cdots}\phi(x^n)/\phi(x^{n-1})=c<+\infty,$

and

(iii) $\int^{+\infty}(xf(\phi(x)))^{-1}dx<+\infty.$

Then, we have

$$\overline{\lim_{s\to+\infty}}\frac{|X(s,\omega)|}{s^H\phi(s)}\leq c \qquad \text{a.s.}$$

Theorem 2. *Let $g(x)$ be a positive, continuous, non-increasing function defined on the positive half line. Assume that $E[g(Y)]$ is finite. Let $\psi(x)$ be a positive, continuous function defined on the positive half line which satisfies the following conditions;*

(i) *$\psi(x)$ is non-increasing,*

and

(ii) $\int^{+\infty}(xg(\psi(x)))^{-1}dx<+\infty.$

Then, we have

$$\underline{\lim_{s\to+\infty}}\frac{\sup_{0\leq t\leq s}|X(t,\omega)|}{s^H\psi(s)}\geq 1 \qquad \text{a.s.}$$

2. First, we prove the following

Lemma 1. *If $E[f(Y)]=K<+\infty$, then for $x>0$, we have*

$$P\Big(\sup_{0\leq t\leq\lambda}|X(t,\omega)|\geq x\Big)\leq K/f(\lambda^{-H}x).$$

Proof. By self-similarity, we have

$$\sup_{0\le t\le \lambda}|X(t,\omega)|\overset{d}{=}\lambda^{H}Y(\omega).$$

Therefore, by Chebyshev's inequality,

$$P\Big(\sup_{0\le t\le \lambda}|X(t,\omega)|\ge x\Big)=P(Y(\omega)\ge \lambda^{-H}x)\le K/f(\lambda^{-H}x).$$

Proof of Theorem 1. Set $x_n=\xi^{nH}\phi(\xi^n)$ for $\xi>1$ and

$$A_n=\Big\{\omega\,;\ \sup_{0\le t\le \xi^n}|X(t,\omega)|\ge x_n\Big\}.$$

Then, by Lemma 1, we have

$$p(A_n)\le K/f(\phi(\xi^n)),$$

and

$$\sum^{\infty}P(A_n)\le C\int^{+\infty}(xf(\phi(x)))^{-1}dx<+\infty.$$

By Borel-Cantelli lemma, there exists $n_0(\omega)<+\infty$ with probability 1 such that

$$\sup_{0\le t\le \xi^n}|X(t,\omega)|\le \xi^{nH}\phi(\xi^n)$$

holds for all $n\ge n_0(\omega)$. Finally, by our assumption for ϕ, we have

$$\frac{|X(s,\omega)|}{s^H\phi(s)}\le\frac{\xi^{nH}\phi(\xi^n)}{\xi^{(n-1)H}\phi(\xi^{n-1})}$$

for $n\ge n_0(\omega)$ and $\xi^{n-1}\le s\le \xi^n$. This concludes that

$$\overline{\lim_{s\to\infty}}\frac{|X(s,\omega)|}{s^H\phi(s)}\le c \qquad \text{a.s.} \qquad \text{Q.E.D.}$$

The proof for Theorem 2 is the same line as that of Theorem 1.

Lemma 2. *If $E[g(Y)]=K'<+\infty$, then for $x>0$ we have*

$$P\Big(\sup_{0\le t\le \lambda}|X(t,\omega)|\le x\Big)\le K'/g(\lambda^{-H}x).$$

Proof of Theorem 2. Set $y_n=\xi^{nH}\psi(\xi^n)$ and

$$B_n=\Big\{\omega\,;\ \sup_{0\le t\le \xi^n}|X(t,\omega)|\le \xi^{nH}\psi(\xi^n)\Big\}.$$

Then, by Lemma 2, we have

$$P(B_n)\le K'/g(\psi(\xi^n)),$$

and

$$\sum^{\infty}P(B_n)\le C'\int^{+\infty}(xg(\psi(x)))^{-1}dx<+\infty.$$

By Borel-Cantelli lemma, there exists $n_1(\omega)<+\infty$ with probability 1 such that

$$\frac{\sup_{0\le t\le s}|X(t,\omega)|}{s^H\psi(s)}\ge\frac{\sup_{0\le t\le \xi^n}|X(t,\omega)|}{\xi^{(n+1)H}\psi(\xi^n)}\ge \xi^{-H}$$

holds for $\xi^n\le s\le \xi^{n+1}$. It follows that

$$\underline{\lim}_{s\to+\infty}\frac{\sup_{0\le t\le s}|X(t,\omega)|}{s^H\psi(s)}\ge 1 \qquad \text{a.s.} \qquad \text{Q.E.D.}$$

3. We can give some examples which are applicable to our theo-

rems; though, results themselves are known ([2]). Let $\{X(t,\omega);\, 0\leq t<+\infty\}$ be a centered path continuous Gaussian self-similar process of order $H>0$ with $E[X(t)^2]=1$. Then, by the theorem due to Fernique [1, Theorem 1.3.3], we have

$$E[e^{(1-\varepsilon)Y^2/2}]<+\infty$$

for $\varepsilon>0$. Therefore, setting $\phi(x)=\sqrt{(2+\delta)\log\log(x+e)}$, we have

$$\int^{+\infty} x^{-1}e^{-(1-\varepsilon)(2+\delta)2^{-1}\log\log x}dx<+\infty \qquad \text{for } \delta>2\varepsilon(1-\varepsilon)^{-1}.$$

Applying Theorem 1, we have

$$\overline{\lim_{s\to+\infty}}\frac{|X(s,\omega)|}{s^H\sqrt{2\log\log s}}\leq 1 \qquad \text{a.s.}$$

To apply Theorem 2, let us consider a narrower class than the previous example. Let $\{X(t,\omega)\}$ be a path continuous Gaussian process with the covariance

$$E[X(t)X(s)]=(|t|^{2H}+|s|^{2H}-|t-s|^{2H})/2$$

for $0<H\leq 1/2$. Then, by [2], we have

$$P(Y\leq y)\leq c_1e^{-c_2y^{-1/H}}.$$

This follows that $E[g(y)]<+\infty$ for $g(x)=\exp\{(c_2-\varepsilon)x^{-1/H}\}$, $\varepsilon>0$. Therefore, setting $\psi(x)=(c_2-2\varepsilon)^H(\log\log x)^{-H}$ we have

$$\int^{+\infty}(xg(\psi(x)))^{-1}dx<+\infty.$$

Applying Theorem 2, we have

$$\underline{\lim}_{s\to+\infty}\frac{(\log\log s)^H\sup_{0\leq t\leq s}|X(t,\omega)|}{s^H}\geq c_2^H, \qquad \text{a.s.}$$

References

[1] Fernique, X.: Regularité des trajéctoires des fonctions aléatoires gaussiennes. Lect. Note in Math., vol. 480, Springer (1975).

[2] Kôno, N.: Evolution asymptotique des temps d'arrêt et des temps de séjour liés aux trajectoires de certaines fonctions aléatoires gaussiennes. Proc. of the Third Japan-USSR Symposium on Probability Theory. ibid., vol. 550, Springer, pp. 290–296 (1976).