19. On Consistency Relations for Polynomial Splines at Mesh and Mid Points

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Let $Q_{p+1}(x)$ be the B-spline defined by

$$Q_{p+1}(x)=(1/p\,!)\sum_{i=0}^{p+1}(-1)^i\binom{p+1}{i}(x-i)_+^p,$$

then we take a polynomial spline function $s(x)$ of the form;

$$(*)\qquad s(x)=\sum_{i=-p}^{n-1}\alpha_i Q_{p+1}(x/h-i),\qquad nh=1$$

with undetermined coefficients α_i, $i=-p,\ -p+1,\cdots,n-1$.

Various consistency relations have been obtained by many authors ([1]–[5]). Here we are concerned with consistency relations at mesh and mid points. If $p=2$, i.e., s is quadratic spline, the following consistency relation is known:

$$(1/8)(s_{i+1}+6s_i+s_{i-1})=(1/2)(s_{i+1/2}+s_{i-1/2})$$

where $s_i=s(ih)$ and $s_{i+1/2}=s((i+1/2)h)$ ([3]).

In the present paper we shall generalize the above relation for polynomial splines of dimensions 1 and 2.

Theorem 1. *Let s be a polynomial spline of the form* $(*)$. *Then we have*

$$h^k(c_0^{(l)}s_i^{(k)}+c_1^{(l)}s_{i+1}^{(k)}+\cdots+c_p^{(l)}s_{i+p}^{(k)})$$
$$=h^l(d_0^{(k)}s_{i+1/2}^{(l)}+d_1^{(k)}s_{i+3/2}^{(l)}+\cdots+d_{p-1}^{(k)}s_{i+p-1/2}^{(l)})$$
$$\text{for } k=0,1,\cdots,p-1 \text{ and } l=0,1,\cdots,p$$

where

$$c_i^{(l)}=Q_{p+1}^{(l)}(p+1/2-i),\qquad d_i^{(k)}=Q_{p+1}^{(k)}(p-i).$$

Proof. Since $Q_{p+1}(x)\equiv 0$ for $x\le 0$ and $x\ge p+1$,

$$c_i^{(l)}=0\qquad \text{for } i\le -1 \text{ and } i\ge p+1$$
$$d_i^{(k)}=0\qquad \text{for } i\le -1 \text{ and } i\ge p.$$

Hence, by substituting $(*)$ into the desired relation, we have

"coefficient of α_j of the left-hand side"

$$=\sum_{m=0}^{p}c_m^{(l)}Q_{p+1}^{(k)}(i+m-j)=\sum_{m=-\infty}^{\infty}Q_{p+1}^{(l)}(p+1/2-m)Q_{p+1}^{(k)}(i+m-j)$$
$$=\sum_{m=-\infty}^{\infty}Q_{p+1}^{(k)}(p-m)Q_{p+1}^{(l)}(i+m+1/2-j)$$

by changing the index,

"coefficient of α_j of the right-hand side"

$$=\sum_{m=0}^{p-1} d_m^{(k)} Q_{p+1}^{(l)}(i+m+1/2-j)$$
$$=\sum_{m=-\infty}^{\infty} Q_{p+1}^{(k)}(p-m) Q_{p+1}^{(l)}(i+m+1/2-j).$$

This completes the proof of this theorem.

As examples of the above relation, let $s(x)$ be a quartic spline, then

$$(1/384)h^k(s_{i+2}^{(k)}+76s_{i+1}^{(k)}+230s_i^{(k)}+76s_{i-1}^{(k)}+s_{i-2}^{(k)})$$
$$=\begin{cases}(1/24)(s_{i+3/2}+11s_{i+1/2}+11s_{i-1/2}+s_{i-3/2}), & k=0\\ (1/6)(s_{i+3/2}+3s_{i+1/2}-3s_{i-1/2}-s_{i-3/2}), & k=1\\ (1/2)(s_{i+3/2}-s_{i+1/2}-s_{i-1/2}+s_{i-3/2}), & k=2\\ (s_{i+3/2}-3s_{i+1/2}+3s_{i-1/2}-s_{i-3/2}), & k=3.\end{cases}$$

These relations are useful for the investigation of the quartic spline interpolation problem at mid points:

$$s_{i+1/2}=f_{i+1/2} \qquad \text{for given function } f(x).$$

Similarly we have the consistency relation for doubly polynomial splines.

Theorem 2. *Let $s(x, y)$ be a doubly polynomial spline function of the form:*

$$s(x, y)=\sum_{i,j=-p}^{n-1} \alpha_{i,j} Q_{p+1}(x/h-i) Q_{p+1}(y/h-j).$$

Then we have

$$h^{l+m}(c_{0,0}^{(k,r)} s_{i,j}^{(l,m)}+c_{0,1}^{(k,r)} s_{i,j+1}^{(l,m)}+\cdots+c_{p,p}^{(k,r)} s_{i+p,j+p}^{(l,m)})$$
$$=h^{k+r}(d_{0,0}^{(l,m)} s_{i+1/2,j+1/2}^{(k,r)}+d_{0,1}^{(l,m)} s_{i+1/2,j+3/2}^{(k,r)}+\cdots+d_{p-1,p-1}^{(l,m)} s_{i+p-1/2,j+p-1/2}^{(k,r)})$$
$$l, m=0,1,\cdots,p-1 \text{ and } k, r=0,1,\cdots,p$$

where

$$s_{i,j}^{(l,m)}=\frac{\partial^{l+m}}{\partial x^l \partial y^m} s(ih, jh)$$
$$s_{i+1/2,j+1/2}^{(k,r)}=\frac{\partial^{k+r}}{\partial x^k \partial y^r} s((i+1/2)h, (j+1/2)h)$$
$$c_{i,j}^{(k,r)}=Q_{p+1}^{(k)}(p+1/2-i) Q_{p+1}^{(r)}(p+1/2-j)$$
$$d_{i,j}^{(l,m)}=Q_{p+1}^{(l)}(p-i) Q_{p+1}^{(m)}(p-j).$$

From above, we have the consistency relation for doubly quadratic spline $s(x, y)$:

$$(1/64)\{s_{i+1,j+1}+s_{i+1,j-1}+s_{i-1,j+1}+s_{i-1,j-1}$$
$$+6(s_{i+1,j}+s_{i,j+1}+s_{i,j-1}+s_{i-1,j})+36s_{i,j}\}$$
$$=(1/4)(s_{i+1/2,j+1/2}+s_{i+1/2,j-1/2}+s_{i-1/2,j+1/2}+s_{i-1/2,j-1/2}).$$

This relation is required for the investigation of the biquadratic spline interpolation at mid points:

$$s_{i+1/2,j+1/2}=f_{i+1/2,j+1/2} \qquad \text{for given function } f(x, y).$$

And we also have the relation which is useful for the construction of the difference scheme for a boundary value problem $\Delta u=f$:

$$(1/4)\{s_{i+1,j+1}+s_{i+1,j-1}+s_{i-1,j+1}+s_{i-1,j-1}$$
$$+2(s_{i+1,j}+s_{i,j+1}+s_{i,j-1}+s_{i-1,j})-12s_{i,j}\}$$

$$=(1/4)h^2(\Delta s_{i+1/2,j+1/2}+\Delta s_{i+1/2,j-1/2}\Delta s_{i-1/2,j+1/2}+\Delta s_{i-1/2,j-1/2}).$$

The discretization error of this nine-point difference scheme is

$$-(1/24)h^4(u_{i,j}^{(4,0)}+u_{i,j}^{(0,4)})+\cdots.$$

On the other hand, those of the central difference scheme and the difference scheme associated with cubic spline collocation method are $(1/12)h^4(u_{i,j}^{(4,0)}+u_{i,j}^{(0,4)})+\cdots$ and $-(1/12)h^4(u_{i,j}^{(4,0)}+u_{i,j}^{(0,4)})+\cdots$, respectively.

In another paper we shall consider the application of this scheme to the numerical solution of the boundary value problem: $\Delta u=f$.

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References

[1] J. Ahlberg, E. Nilson, and J. Walsh: The Theory of Splines and their Applications. Academic Press, New York (1967).

[2] W. Hoskins and D. Meek: Linear dependence relations for polynomial splines at midpoints. BIT, **15**, 272–276 (1975).

[3] W. Kammerer, G. Reddien, and R. Varga: Quadratic splines. Numer. Math., **22**, 241–259 (1974).

[4] T. Lucas: Error bounds for interpolating cubic splines under various end conditions. SIAM J. Numer. Anal., **11**, 569–584 (1974).

[5] M. Sakai: Some new linear relations for odd degree polynomial splines at mid-points. Proc. Japan Acad., **59**A, 24–25 (1983).