128. On q-Additive Functions. II

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1. Let q be an arbitrary fixed natural number $\geqslant 2$, and $g(n)$ be a q-additive arithmetical function. In our previous paper [2], we proved a functional equation involving a q-additive function. The purpose of this article is to prove a general theorem which gives explicitly an average value of some q-additive functions as an application of our previous result.

2. In the first place, we shall mention two simple special cases of our Theorem, to clarify its nature.

(I) The relation $g(rq^k)=r$, $1 \leqslant r \leqslant q-1$, $k \in N$, defines a q-additive function $g(n)$ called "sum of digits", treated in [1]. For this function, we have

$$\frac{1}{m} \sum_{n=0}^{m-1} g(n) = \frac{q-1}{2(\log q)} \log m + F\left(\frac{\log m}{\log q}\right),$$

for any $m \in N$, where $F(x)$ is a periodic function with period 1 and its Fourier coefficients are given as follows:

$$F(x) = \sum_{k \in Z} A_k \cdot \exp(2\pi i k x),$$

$$A_0 = \frac{q-1}{2(\log q)} \{(\log 2\pi) - 1\} - \frac{q+1}{4},$$

$$A_k = i \frac{q-1}{2\pi k} \left(\frac{2\pi i k}{\log q} + 1\right)^{-1} \cdot \zeta\left(\frac{2\pi i k}{\log q}\right), \qquad k \neq 0.$$

(II) We define a 2-additive function $g(n)$ by the relation $g(2^k)=k$ for $k \in N$. Then we have

$$\frac{1}{m} \sum_{n=0}^{m-1} g(n) = \frac{1}{4(\log^2 2)} \log^2 m + (\log m) F_1\left(\frac{\log m}{\log 2}\right) + F_2\left(\frac{\log m}{\log 2}\right),$$

where $F_1(x)$ and $F_2(x)$ are periodic functions with period 1 and whose Fourier expansions are given as follows:

$$F_1(x) = \sum_{k \in Z} c_k \cdot \exp(2\pi i k x), \qquad F_2(x) = \sum_{k \in Z} d_k \cdot \exp(2\pi i k x),$$

$$c_0 = \frac{(\log 2\pi) - 1}{2(\log^2 2)} - \frac{1}{\log 2},$$

$$c_k = \frac{i}{2k\pi(\log 2)} \cdot \zeta\left(\frac{2\pi i k}{\log 2}\right) \cdot \left(\frac{2\pi i k}{\log 2} + 1\right)^{-1}, \qquad k \neq 0,$$

$$d_0 = \frac{11}{24} - \frac{\zeta''(0)}{\log^2 2} - \frac{(\log 2\pi) - 1}{\log 2} - \frac{(\log 2\pi) - 1}{2\log^2 2},$$

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$$d_k = \frac{1}{2\pi i k \, (\log 2)} \left(\frac{2\pi i k}{\log 2} + 1 \right)^{-1}$$

$$\times \left[\left\{ 2 \log 2 + \frac{\log 2}{2\pi i k} + \left(\frac{2\pi i k}{\log 2} + 1 \right)^{-1} \right\} \cdot \zeta\left(\frac{2\pi i k}{\log 2} \right) - \zeta'\left(\frac{2\pi i k}{\log 2} \right) \right], \quad k \neq 0.$$

3. In order to state our result, we introduce some notations. Let $g(n)$ be a q-additive function. We put

(1) $f_r(s) = \sum_{k=0}^{\infty} g(rq^k) q^{-(k+1)s}$, $1 \leqslant r \leqslant q-1$,

and we suppose that $f_r(s)$, $1 \leqslant r \leqslant q-1$, are rational functions of q^s. Then any pole of $f_r(s)$ is isolated and the cardinal number of the set

$$\Pi_r = \{\rho : \rho \text{ is a pole of } f_r(s), \ 0 \leqslant \mathrm{Im}\,(\rho) < 2\pi/(\log q)\},$$

is finite. We define the numbers $d(r, \rho)$ and $C_{r,\rho}(n)$ for $1 \leqslant r \leqslant q-1$, $\rho \in \Pi_r$, by Laurent expansion of $f_r(s)$ at $s = \rho$:

$$f_r(s) = \sum_{n=-d(r,\rho)}^{\infty} C_{r,\rho}(n) \cdot (s-\rho)^n.$$

Since $f_r(s)$ can also be expanded into Dirichlet series in the half plane $\mathrm{Re}\,(s) < \mathrm{Min}_{\rho \in \Pi_r} \{\mathrm{Re}\,(\rho)\}$, we can define the numbers $u(r)$ and $E_r(n)$ by

(2) $f_r(s) = \sum_{n=-u(r)}^{\infty} E_r(n) \cdot q^{ns}$, $1 \leqslant r \leqslant q-1$.

In addition, we make use of the following symbols:

$[y] =$ the largest integer not exceeding y, $\langle y \rangle = y - [y]$,

$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$, $0 < a \leqslant 1$,   (Hurwitz zeta function),

$\zeta^{(h)}(s, a) = (d^h/ds^h)\zeta(s, a)$, $h \in N$.

Then our main theorem states:

Theorem. *Let $g(n)$ be a q-additive function. Suppose all the functions $f_r(s)$, $1 \leqslant r \leqslant q-1$, defined by (1) satisfy the two conditions:*

  i)   *$f_r(s)$ are rational functions of q^s,*

  ii) *any pole of $f_r(s)$ is contained in the half plane $\mathrm{Re}\,(s) > -1/2$.*

Then the set Π_r, the numbers $d(r, \rho)$, $C_{r,\rho}(n)$, $u(r)$ and $E_r(n)$ are defined as above. Then we have

(♯) $m^{-1} \sum_{n=0}^{m-1} g(n) = \sum_{r=1}^{q-1} \{A_r (\log m)^{d(r,0)} + B_r + W_r(m)\}$

$$+ \sum_{r=1}^{q-1} \sum_{\rho \in \Pi_r} \sum_{l=0}^{d(r,\rho)-1} m^\rho (\log m)^l \cdot F_{r,\rho,l}\left(\frac{\log m}{\log q} \right),$$

for any $m \in N$, where

$$A_r = \begin{cases} q^{-1} C_{r,0}(0)/d(r,0)! & \text{if } s=0 \text{ is a pole of } f_r(s), \\ 0 & \text{otherwise}, \end{cases}$$

$$B_r = \begin{cases} 0 & \text{if } s=0 \text{ is a pole of } f_r(s), \\ q^{-1} f_r(0) & \text{otherwise}, \end{cases}$$

$$W_r(m) = \sum_{n=-u(r)}^{-1} E_r(n) \cdot w_r(n, m), \quad \text{with}$$

(∗) $w_r(n, m) = \begin{cases} \dfrac{1}{m} q^{n+1}(1 - \langle q^n m \rangle), & \text{if } \langle q^n m \rangle \geqslant \dfrac{r+1}{q}, \\[2mm] \dfrac{1}{m} q^{n+1}\{(q-1) \cdot \langle q^n m \rangle - r\}, & \text{if } \dfrac{r+1}{q} > \langle q^n m \rangle \geqslant \dfrac{r}{q}, \\[2mm] \dfrac{1}{m} q^{n+1} \cdot \langle q^n m \rangle, & \text{if } \dfrac{r}{q} > \langle q^n m \rangle. \end{cases}$

*Furthermore every function $F_{r,\rho,l}(x)$ is a periodic function with period
1 and its Fourier coefficients are given explicitly as follows:*

$$F_{r,\rho,l}(x)=\sum_{k\in Z}A_{r,\rho,l}(k)\cdot\exp(2\pi ikx)$$

with

$$A_{r,0,l}(0)=\frac{1}{l!}\sum_{\substack{i+j+h=\bar{d}(r,0)-l\\i,j,h\geqslant0}}\frac{(-1)^j}{h!}C_{r,0}(i)\left\{\zeta^{(h)}\left(0,\frac{r}{q}\right)-\zeta^{(h)}\left(0,\frac{r+1}{q}\right)\right\},$$

$$A_{r,1,l}(0)=\frac{1}{l!}\sum_{\substack{i+j+h=\bar{d}(r,1)-l\\i,j,h\geqslant0}}\frac{(-1)^j}{h!}C_{r,1}(i)\cdot(1-2^{-(j+1)})$$

$$\times\left\{\lim_{s\to1}\frac{d^h}{ds^h}\left[(s-1)\left(\zeta\left(s,\frac{r}{q}\right)-\zeta\left(s,\frac{r+1}{q}\right)\right)\right]\right\},$$

$$A_{r,\rho,l}(k)=\frac{1}{l!}\sum_{\substack{i+j+h=\bar{d}(r,\rho)-l-1\\i,j,h\geqslant0}}\frac{(-1)^j}{h!}C_{r,\rho}(i)$$

$$\times\left\{\left(\frac{2\pi ik}{\log q}+\rho\right)^{-(j+1)}-\left(\frac{2\pi ik}{\log q}+\rho+1\right)^{-(j+1)}\right\}$$

$$\times\left\{\zeta^{(h)}\left(\frac{2\pi ik}{\log q}+\rho,\frac{r}{q}\right)-\zeta^{(h)}\left(\frac{2\pi ik}{\log q}+\rho,\frac{r+1}{q}\right)\right\},\begin{pmatrix}k\neq0\ \ or\\\rho\neq0,\ 1\end{pmatrix}.$$

4. *Sketch of proof.* 1) We start from the functional equation
which was given in [2],

$$s\int_1^\infty g([t])t^{-s-1}dt=\sum_{r=1}^{q-1}\left\{\zeta\left(s,\frac{r}{q}\right)-\zeta\left(s,\frac{r+1}{q}\right)\right\}\cdot f_r(s).$$

By making use of convolution product of Laplace transform, we get

$$\int_0^\infty G(y)e^{-sy}dy=\sum_{r=1}^{q-1}\left\{\zeta\left(s,\frac{r}{q}\right)-\zeta\left(s,\frac{r+1}{q}\right)\right\}\frac{f_r(s)}{s(s+1)},\qquad\mathrm{Re}\,(s)>\alpha_0,$$

where $G(y)=e^{-y}\int_0^y g([e^w])e^w dw$, and $\alpha_0=\mathrm{Max}_{1\leqslant r\leqslant q-1}$ {the convergence
abscissa of $f_r(s)$}.   Note that if $y=m\in N$, $G(m)=m^{-1}\sum_{n=0}^{m-1}g(n)$.

2)   By means of Laplace inverse formula, we have

$$G(m)=\sum_{r=1}^{q-1}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left\{\zeta\left(s,\frac{r}{q}\right)-\zeta\left(s,\frac{r+1}{q}\right)\right\}\frac{f_r(s)}{s(s+1)}m^s ds,\qquad\alpha>\alpha_0.$$

From the condition ii) of our theorem, we can take a negative number
η satisfying $-1/2<\eta<\mathrm{Re}\,(\rho)$ for any $\rho\in\bigcup_{r=1}^{q-1}\Pi_r$.   Now we shift the
contour of the above integral to the line $\mathrm{Re}\,(s)=\eta$.   By Cauchy's
integral theorem, we obtain

(3) $$G(m)=\sum_{r=1}^{q-1}\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\left\{\zeta\left(s,\frac{r}{q}\right)-\zeta\left(s,\frac{r+1}{q}\right)\right\}\frac{f_r(s)}{s(s+1)}m^s ds$$

$$+\sum_{r=1}^{q-1}\sum_{\rho\in\Pi_r}\sum_{k\in Z}\left(\text{residue of the integrand at }s=\rho+\frac{2\pi ik}{\log q}\right).$$

Here we have to prove the two facts:

    a)   *the convergence of the sum of residues,*

    b) $$\lim_{n\to\infty}\int_{\alpha+iT_n}^{\eta+iT_n}\left\{\zeta\left(s,\frac{r}{q}\right)-\zeta\left(s,\frac{r+1}{q}\right)\right\}\frac{f_r(s)}{s(s+1)}m^s ds=0,$$

where $\{T_n\}$ is a suitably chosen sequence satisfying $\lim_{n\to\infty}T_n=\infty$.

We can prove a), b) using the following lemma:

Lemma 1. *Let h be a non-negative integer, and β, T_0 be fixed positive numbers. If $\sigma > -\beta$ and $|t| > T_0$, then, for any $\varepsilon > 0$,*

$$\frac{d^h}{ds^h}\zeta(\sigma + it, a) = O(|t|^{\beta + (1/2) + \varepsilon}),$$

and the constant implied by O-symbol depends on h, a, β and T_0.

The right hand side of (3) consists of the "integral part" and the "residue part". In calculating the residues, we obtain all terms of the right hand side of (♯) except the terms $W_r(m)$. These last terms come from the "integral part".

3) From the condition ii) of our Theorem, the expansion (2) converges absolutely on the line $\mathrm{Re}(s) = \eta$. By virtue of Lebesgue's convergence theorem we obtain

$$(4) \quad \frac{1}{2\pi i}\int_{\eta - i\infty}^{\eta + i\infty}\left\{\zeta\left(s, \frac{r}{q}\right) - \zeta\left(s, \frac{r+1}{q}\right)\right\}\frac{f_r(s)}{s(s+1)}m^s ds$$

$$= \sum_{n=-u(r)}^{\infty} E_r(n)\frac{1}{2\pi i}\int_{\eta - i\infty}^{\eta + i\infty}\left\{\zeta\left(s, \frac{r}{q}\right) - \zeta\left(s, \frac{r+1}{q}\right)\right\}\frac{1}{s(s+1)}(q^n m)^s ds.$$

In order to calculate these integrals, we need the following lemma.

Lemma 2. *For any positive γ satisfying $-1/2 < -\gamma < 0$, and for any a such that $0 < a \leqslant 1$, we have*

$$\frac{1}{2\pi i}\int_{-\gamma - i\infty}^{-\gamma + i\infty}\zeta(s, a)\frac{x^s}{s(s+1)}ds = a + \frac{[x-a]+1}{x}\{(x-a)-[x-a]\}$$

$$+ (2x)^{-1}\{([x-a]+2)([x-a]-1)-(x+2)(x-1)\}.$$

Using this lemma, firstly we can show that the integral terms with $n \geqslant 0$ in the right hand side of (4) are equal to zero. And secondly, $w_r(n, m)$ being the value of the integral term with $n < 0$, we get (∗). We denote the value of the formula (4), which is expressed by a finite sum, by $W_r(m)$. Then we obtain the desired formula (♯).

References

[1] H. Delange: Sur la fonction sommatoire de la fonction "somme de chiffres". L'Enseignement Math., **21**, 31–47 (1974).

[2] J.-L. Mauclaire and Leo Murata: On q-additive functions. I. Proc. Japan Acad., **59A**, 274–276 (1983).