125. Geometry of Yang-Mills Connections over a Kähler Surface

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1. In [2] and [3] we showed that the moduli space of irreducible anti-self-dual Yang-Mills connections over a compact Kähler surface with positive scalar curvature has a smooth manifold structure. In this paper we exhibit theorems with brief proofs about a complex structure on the moduli space of anti-self-dual connections.

Let A be an anti-self-dual connection on a principal bundle P with positive second Chern class over a Kähler surface. Then for the adjoint bundle $\mathfrak{g}_P$ the sequence

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^2_+(\mathfrak{g}_P) \longrightarrow 0$$

defines an elliptic complex and A induces a holomorphic structure on g_P^C compatible with d_A. We call A generic when the 0-th cohomology group H^0 and the second cohomology group H^2 of the above sequence vanish.

Theorem 1. *The moduli space $\mathcal{M}_0$ of generic anti-self-dual connections over a compact Kähler surface has a complex structure.*

Theorem 2. *Let M be a compact Kähler surface either with trivial canonical line bundle K_M or with positive total scalar curvature. Then the moduli space of irreducible* (i.e., $H^0=0$) *anti-self-dual connections over M is a complex manifold.*

A (0,1)-connection $\tilde{A}$ satisfying integrability condition $\bar{\partial}\tilde{A}-\tilde{A}\wedge\tilde{A}=0$ is called holomorphic. With respect to the moduli space $\mathcal{M}_h$ of holomorphic connections we have

Theorem 3. *The moduli space $(\mathcal{M}_h)_0$ of generic, holomorphic connections on an $SU(n)$-principal bundle has a complex structure of complex dimension $c_2(\mathfrak{g}_P^C)-\dim SU(n)\cdot p_a(M)$, if it is not empty. Moreover the canonical map h of $\mathcal{M}_0$ to $(\mathcal{M}_h)_0$ is holomorphic and of maximal rank, and the image of $\mathcal{M}_0$ is open and closed in $(\mathcal{M}_h)_0$.*

It is easily observed that there is a one-to-one correspondence between the moduli $\mathcal{M}_h$ and moduli $\mathcal{M}_J$ of holomorphic structures on $\mathfrak{g}_P^C$. It is known on the other hand that over a nonsingular projective surface $\mathcal{M}_J$ on a rank two vector bundle is a quasi-projective variety [7]. From the sufficient evidence that $\mathcal{M}$ has even dimension and each irreducible anti-self-dual connections induces a stable holomorphic structure on $\mathfrak{g}_P^C$ [6] it is conjectured by Atiyah that $\mathcal{M}$ must be a complex manifold [1].

2. Anti-self-dual connection and holomorphic connection. Let P be an $SU(n)$-principal bundle with adjoint bundle $\mathfrak{g}_P$. A connection A on P with anti-self-dual curvature form $F(A)=dA-A\wedge A$ is called also anti-self-dual. The group of gauge transformations $\mathcal{G}$ acts on the set of connections $\mathcal{C}$ and leaves the subset of anti-self-dual connections invariant. Hence we have its quotient, called moduli $\mathcal{M}$ of anti-self-dual connections on P. A connection A is irreducible (reducible) if $d_A:\Omega^0(\mathfrak{g}_P)\to\Omega^1(\mathfrak{g}_P)$ has zero kernel (non-zero kernel).

A connection $\tilde{A}$ which induces a covariant derivative of type (0, 1) $\bar{\partial}_{\tilde{A}}:\Omega^0(\mathfrak{g}_P^C)\to\Omega^{0,1}(\mathfrak{g}_P^C)$ is called (0, 1)-connection. We call (0, 1)-connection $\tilde{A}$ holomorphic if its curvature $F(\tilde{A})=\bar{\partial}\tilde{A}-\tilde{A}\wedge\tilde{A}$ vanishes. The group of complex gauge transformations $\mathcal{G}^C$ acts on the set of (0, 1)-connections $\mathcal{C}^{(0,1)}$ and induces moduli space $\mathcal{M}_h$ of holomorphic connections.

For each A in $\mathcal{C}$ the (0, 1)-component $A^{0,1}$ is in $\mathcal{C}^{(0,1)}$ and further $A^{0,1}$ is holomorphic for anti-self-dual A. Conversely $A=\tilde{A}-{}^t\overline{\tilde{A}}$ belongs to $\mathcal{C}$ for each $\tilde{A}\in\mathcal{C}^{(0,1)}$. Since $\mathcal{G}\subset\mathcal{G}^C$ we have a canonical map h from $\mathcal{M}$ to $\mathcal{M}_h$.

For each holomorphic connection $\tilde{A}$ we have the elliptic complex

$$0\longrightarrow\Omega^0(\mathfrak{g}_P^C)\xrightarrow{\bar{\partial}_{\tilde{A}}}\Omega^{0,1}(\mathfrak{g}_P^C)\xrightarrow{\bar{\partial}_{\tilde{A}}}\Omega^{0,2}(\mathfrak{g}_P^C)\longrightarrow 0$$

with k-th cohomology group H^k. By using the Atiyah-Singer index theorem the index is given by $-c_2(\mathfrak{g}_P^C)+\dim SU(n)\cdot p_a(M)$. We call holomorphic connection generic if $H^0=H^2=0$.

3. Brief proofs of theorems. Lemma 1. *For each irreducible* $[\tilde{A}]\in\mathcal{M}_h$, $V_{\tilde{A}}=\{\alpha\in\Omega^{0,1}(\mathfrak{g}_P^C)\,;\,\|\alpha\|<\varepsilon,\ \bar{\partial}_{\tilde{A}}^*\alpha=0,\ \bar{\partial}_{\tilde{A}}\alpha=\alpha\wedge\alpha\}$ *gives a neighborhood of* $[\tilde{A}]$ *in* $\mathcal{M}_h$.

Let $\Psi=\Psi_{\tilde{A}}:\Omega^{0,1}(\mathfrak{g}_P^C)\to\Omega^{0,1}(\mathfrak{g}_P^C)$ be given by $\alpha\to\alpha-\bar{\partial}_{\tilde{A}*}G(\alpha\wedge\alpha)$, where G is the Green operator. Then we have

Lemma 2. $\Psi(V_{\tilde{A}})\subset H^1_{\tilde{A}}$ *and* Ψ *has an inverse over an* ε*-neighborhood* $V_\varepsilon\subset H^1_{\tilde{A}}$ *and moreover* $\Psi^{-1}|_{V_\varepsilon}$ *is holomorphic.*

To show that each local charts $\Psi_{\tilde{A}}:V_{\tilde{A}}\to V_\varepsilon$ are holomorphically related we require the following lemma. For each $[\tilde{A}_1],[\tilde{A}_2]\in(\mathcal{M}_h)_0$ satisfying $\pi(\tilde{A}_1+V_{\tilde{A}_1})\cap\pi(\tilde{A}_2+V_{\tilde{A}_2})\neq\phi$ we have that for $\alpha\in V_{\tilde{A}_1}$ there exists a unique $f=f_\alpha\in\mathcal{G}^C$ such that $f(\tilde{A}_1+\alpha)\in\tilde{A}_2+V_{\tilde{A}_2}$.

Lemma 3. f_α *depends holomorphically on* t, *if* $\alpha=\Psi^{-1}(t)$ *is parametrized by complex coordinates* t *of* $H^1_{\tilde{A}}$.

This is based on the standard fact that solutions of a quasi-linear elliptic equation with parameters of holomorphic functions of t depend holomorphically on t. By combining these lemmas we obtain the first part of Theorem 3.

Now we let A be an anti-self-dual connection. Then we have by using the Bochner-Weitzenböck formula of the operator $d_A^+ \circ d_{A*}^+$ the following

Lemma 4. *H_A^2 is $\boldsymbol{R}$-isomorphic to $H_A^0 \oplus \boldsymbol{H}$, where $\boldsymbol{H}$ denotes $H^0(M; \mathcal{O}(\mathfrak{g}_P^{\boldsymbol{C}} \otimes K_M))$ with respect to the holomorphic structure of $\mathfrak{g}_P^{\boldsymbol{C}}$ induced from A, and $H_{\tilde{A}}^2 \cong \boldsymbol{H}$ for $\tilde{A} = A^{0,1}$.*

Lemma 5. *If A is irreducible, then so is the holomorphic connection $\tilde{A} = A^{0,1}$.*

From these lemmas we have $h(\mathcal{M}_0) \subset (\mathcal{M}_h)_0$. By applying a local slice neighborhood argument we can show that h is a local diffeomorphism. Since $\langle F(A), \omega_g \rangle = 0$ for each $[A] \in \mathcal{M}$, the image $h(\mathcal{M}_0)$ is closed in $(\mathcal{M}_h)_0$. Thus we obtain Theorem 3 and also automatically Theorem 1. Theorem 2 is an easy consequence of Theorem 1 and Lemma 4.

4. Remark. Each $[A] \in \mathcal{M}$ which is not generic gives a singularity of $\mathcal{M}$ [5]. Let L be a nontrivial holomorphic line bundle with $c_1(L) \wedge [\omega_g] = 0$. Then an $SU(2)$-bundle P associated to $L \oplus L^{-1}$ admits a reducible anti-self-dual connection A. The moduli space $\mathcal{M}$ is of course singular at $[A]$. Singular points near $[A]$ consist of a $b_1(M)$-dimensional open ball, if the base space is of positive total scalar curvature. Moreover the bundle P also admits irreducible, hence generic anti-self-dual connections near $[A]$.

Detailed proofs of these theorems and lemmas will be given in a forthcoming paper [4].

References

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