124. On Geodesic with the Same Angle*)

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1. Introduction. Let M be a surface of revolution with the parametrization $f(u,v)=(a(u)\cos v,\ a(u)\sin v,\ b(v))$. Let α be a geodesic on M, $r(t)$ the radius of the parallel ($u=\text{const.}$) and $\theta(t)$ the angle between α and the meridium ($v=\text{const.}$) at $\alpha(t)$. Then the relation

$$r(t)\cdot\sin\theta(t)=\text{const.}\qquad \text{at } \alpha(t)$$

is satisfied for any geodesic α on M. This is "Clairaut's theorem for a surface of revolution". This theorem implies that for some parallel submanifold N ($u=\text{const.}$) there exists a geodesic α on M acrossing N with the same angle: that is for some intersection points $\alpha(t_1), \alpha(t_2) \in N\cap\alpha$, $\sphericalangle(\alpha(t_1), T_{\alpha(t_1)}N)=\sphericalangle(\alpha(t_2), T_{\alpha(t_2)}N)$ where $\sphericalangle(\alpha(t_i), T_{\alpha(t_i)}N)$ is the angle between $\alpha(t_i)$ and the 1-dimensional line $T_{\alpha(t_i)}N$. This geodesic is called "a geodesic with the same angle for N". (The definition will be given later. Roughly speaking it is a geodesic on M which starts from the submanifold N with one angle and ends in N with the same angle.) In our previous paper [3] we considered under the condition of codimension 1 the existence of such geodesics which is a higher dimensional generalization of the above theorem in a sense. In this note we consider the general case of an arbitrary codimension. Our main result is the following

Theorem. *Let M be a complete Riemannian manifold, N a connected compact closed submanifold without boundary and f an isometry on M with $f(N)=N$.*

If f has a finite number of fixed points on N and if M is simply connected, then there exists a non-trivial geodesic with the same angle for N, except the case in which f has exactly one fixed point.

If f is fixed point free, then there exists always a non-trivial geodesic with the same angle for N.

Our method of proof is much the same as our previous paper. The author would like to thank Prof. M. Tanaka of Tokai University for his advice to remove the condition of codimension 1 and his encouragement.

2. Hilbert submanifolds and energy functions. Let M be a complete Riemannian manifold with a Riemannian metric $\langle\ ,\ \rangle$ and let $L_1^2(I,M)$ be the set of absolutely continuous maps from the unit

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interval $I=[0,1]$ to M with square integrable derivative. Then it is known that $L_1^2(I, M)$ is a complete Hilbert Riemannian manifold with the Riemannian structure given by

$$\langle\!\langle X_\alpha, Y_\alpha \rangle\!\rangle = \int_0^1 \{\langle X_\alpha(t), Y_\alpha(t)\rangle + \langle \nabla_\alpha X_\alpha(t), \nabla_\alpha Y_\alpha(t)\rangle\} dt$$

where X_α, Y_α are elements of the tangent space $T_\alpha L_1^2(I, M)$ at α of $L_1^2(I, M)$ which is a linear space of absolutely continuous vector field along α on M with square integrable covariant derivative $\nabla_\alpha X_\alpha$. Let N be a closed submanifold of M without boundary, let f be an isometry on M with $f(N)=N$ and $\Lambda_N(M, f)=\{\alpha \in L_1^2(I, M) \mid f(\alpha(0))=\alpha(1), \alpha(0) \in N\}$, then we have the following

Proposition 2-1 [3]. *$\Lambda_N(M, f)$ is a Hilbert Riemannian submanifold of $L_1^2(I, M)$ and its tangent space $T_\alpha \Lambda_N(M, f)$ at α is given by*

$$T_\alpha \Lambda_N(M, f) = \{X_\alpha \in T_\alpha L_1^2(I, M) \mid (X_\alpha(0), X_\alpha(1)) \in T_{\alpha(0)}N \times T_{\alpha(1)}N,\ f_* X_\alpha(0) = X_\alpha(1)\}.$$

Define an energy function $E : L_1^2(I, M) \to R$ by

$$E(\alpha) = \frac{1}{2}\int_0^1 \|\dot\alpha(t)\|^2\, dt.$$

Then it is a C^∞ map. In particular we consider the restriction of E to $\Lambda_N(M, f)$ and denote it by the same symbol E. Since a critical point for E is a geodesic, we have only to study a critical point for E. In fact the following general proposition has been obtained by K. Grove.

Proposition 2-2 [1]. *If V is a submanifold of $L_1^2(I, M)$ such that $T_\alpha V$ for $\alpha \in V$ contains all $X_\alpha \in T_\alpha L_1^2(I, M)$ with $X_\alpha(0)=0$ and $X_\alpha(1)=0$ and if α is a critical point for $E|_V$, then α is C^∞ and α is a geodesic.*

Now we introduce a new type of geodesics in order to see what sort of geodesic is a critical point of E on $\Lambda_N(M, f)$. Let M be a complete Riemannian manifold and let N be a closed submanifold. Let $c : I \to M$ be a smooth curve with $x=c(t) \in N$ for some $t \in I$ and let P be an orthogonal projection of T_xM to T_xN. We denote the angle between $\dot c(t)$ and $P(\dot c(t))$ by $\measuredangle(\dot c(t), T_xN)$.

Definition 2-3. A geodesic $\alpha : I \to M$ with $(\alpha(0), \alpha(1)) \in N \times N$ is called "a geodesic with the same angle for N" (simply "a geodesic with the same angle") if $\measuredangle(\dot\alpha(0), T_{\alpha(0)}N) = \measuredangle(\dot\alpha(1), T_{\alpha(1)}N)$.

In our previous paper we introduced the concept of the same angle which was defined by the angle between the velocity vector of α and the normal vector of N in M in view of the condition of codimension 1. However we can find that the theory developed in [3] is valid for our new concept of the same angle. The following theorem is essential in order to get a satisfactory development of our previous theory. Combining Propositions 2-1 and 2-2, we have

Theorem 2-4. *Let M be a complete Riemannian manifold, N a* *closed submanifold without boundary and f an isometry of M with $f(N)=N$.*

If $\alpha \in \Lambda_N(M, f)$ is a critical point for $E: \Lambda_N(M, f) \to R$, then α is a geodesic with the same angle.

Proof. First we note that

$$dE_\alpha(X_\alpha)=\int_0^1 \langle \nabla_\alpha X_\alpha(t), \dot\alpha(t)\rangle dt$$

for all $X_\alpha \in T_\alpha \Lambda_N(M, f)$ in K. Grove [1]. Assume that $\alpha \in \Lambda_N(M, f)$ is a critical point of E, then α is a geodesic on M by Proposition 2-2 so that for any tangent vector $X_\alpha \in T_\alpha \Lambda_N(M, f)$

$$\begin{aligned} dE_\alpha(X_\alpha) &= \int_0^1 \langle \nabla_\alpha X_\alpha(t), \dot\alpha(t)\rangle dt = \int_0^1 \{\langle \nabla_\alpha X_\alpha(t), \dot\alpha(t)\rangle + \langle X_\alpha(t), \nabla\dot\alpha(t)\rangle\} dt \\ &= \int_0^1 \frac{d}{dt}\langle X_\alpha(t), \dot\alpha(t)\rangle dt = \langle X_\alpha(1), \dot\alpha(t)\rangle - \langle X_\alpha(0), \dot\alpha(0)\rangle \\ &= \langle X_\alpha(1), \dot\alpha(1) - f_*\dot\alpha(0)\rangle. \end{aligned}$$

Since α is a critical point of E, we have $\dot\alpha(1)-f_*\dot\alpha(0) \perp T_{\alpha(1)}N$ and therefore $P_1(\dot\alpha(1))=P_1(f_*\dot\alpha(0))$. On the other hand, since

$$(f|_N)_*: T_{\alpha(0)}N \to T_{\alpha(1)}N$$

is a linear isomorphism by the assumption $f(N)=N$ and $P_1 \cdot f_* =(f|_N)_* \cdot P_0$, we have $P_1(\dot\alpha(1))=f_*P_0(\dot\alpha(0))$. Thus

$$0=\langle \dot\alpha(1)-f_*\dot\alpha(0), f_*P_0(\dot\alpha(0))\rangle$$

implies $\langle \dot\alpha(1), P_1(\dot\alpha(1))\rangle = \langle \dot\alpha(0), P_0(\dot\alpha(0))\rangle$. Hence α is a geodesic with the same angle. Q.E.D.

3. Proof of Main Theorem. We can prove the following lemma in the same manner in the proof of Lemma 2-6 [3].

Lemma 3-1. *Let M be a complete Riemannian manifold, N a connected compact closed submanifold without boundary and f an isometry of M such that $f(N)=N$ and $F(N, f)\neq\phi$ where $F(N, f)$ is the set of fixed point of $f|_N$.*

If there exists no non-trivial geodesic with the same angle, then the inclusion $i: F(N, f) \to \Lambda_N(M, f)$ is a homotopy equivalence. Moreover there exists a non-trivial geodesic with the same angle if $F(N, f) =\phi$.

Now we prove the main theorem. If f is fixed point free, then the existence of our geodesic is evident by the last half of Lemma 3-1. So we suppose that f has fixed points on N, that the number of fixed points is finite and is not less than two and that $\pi_1(M)=0$. Since the fibration

$$\Lambda_N(M, f) \to G(M, N, f) \to N$$

has a fiber $\Lambda_p(M)=\{\alpha \in \Lambda_N(M, f) \mid \alpha(0)=\alpha(1)=p\}$ where $f(p)=p$ is a base point in N and $G(M, N, f)=\{(x, f(x)) \mid x \in N\}$, the homotopy exact sequence

$$
\begin{array}{l}
\rightarrow \pi_0(\Lambda_p(M)) \rightarrow \pi_0(\Lambda_N(M, f)) \rightarrow \pi_0(N)=0 \\
\quad \wr\| \\
\pi_0(\Omega_p(M)) \cong \pi_1(M)=0
\end{array}
$$

gives $\pi_0(\Lambda_N(M, f))=0$ because $\Lambda_p(M)$ is the same homotopy type to the loop space $\Omega_p(M)$ at p. By the assumption $\pi_0(F(N, f)) \neq 0$ and we have the conclusion of the main theorem by Lemma 3-1.

References

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