# 122. Generalized Splitting Theorem for Map-Germs

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§1. **Introduction.** Let $\boldsymbol{K}$ be the field of real numbers $\boldsymbol{R}$ or complex numbers $\boldsymbol{C}$. In the classification theory of singularities of $C^\infty$-(resp. holomorphic) functions $f:(\boldsymbol{K}^n, 0)\to(\boldsymbol{K}, 0)$, Thom's splitting theorem plays an important role (see [1], [7]). It can be stated that a $C^\infty$-(resp. holomorphic) function $f$ of $n$ variables whose first derivatives vanish at the origin and the rank of Hessian matrix $(\partial^2 f/\partial x_i\partial x_j)(0)$ equals $n-j$, can be written (after a change of coordinates) in the form $f(x_1, \cdots, x_n)=g(x_1, \cdots, x_j)+\sum_{i=j+1}^n \pm x_i^2$ where the second derivatives of $g$ vanish at the origin. Thus the classification of functions $f(x_1, \cdots, x_n)$ is reduced to that of functions $g(x_1, \cdots, x_j)$.

In this note we shall announce some results on normal forms and generalized splitting theorem for map-germs $f:(\boldsymbol{K}^n, 0)\to(\boldsymbol{K}^p, 0)$ whose rank of Jacobian matrix at the origin equals $p-1$ $(n\geqq p)$. As applications of these theorems we obtain some classification and normal forms of certain finitely determined singularities of map-germs. Details and the applications of these theorems will appear elsewhere.

§2. **Definitions and notations.** Let $\mathcal{E}(n, p)$ be the set of all $C^\infty$-(resp. holomorphic) map-germs $f:(\boldsymbol{K}^n, 0)\to(\boldsymbol{K}^p, 0)$. We denote by $L(n)$ (resp. $L(p)$) the group of ($C^\infty$ or holomorphic) local diffeomorphisms $\varphi:(\boldsymbol{K}^n, 0)\to(\boldsymbol{K}^n, 0)$ (resp. $\psi:(\boldsymbol{K}^p, 0)\to(\boldsymbol{K}^p, 0)$). The group $L(n)\times L(p)$ acts on $\mathcal{E}(n, p)$ as follows; $(\varphi, \psi)f=\psi\circ f\circ\varphi$ where $(\varphi, \psi)\in L(n)\times L(p)$ and $f\in\mathcal{E}(n, p)$.

**Definition 1.** For two map-germs $f, g\in\mathcal{E}(n, p)$, we say that $f$ and $g$ are *k-jet equivalent* if the all partial derivatives of order $\leqq k$ at the origin of $f$ and $g$ are equal.

**Definition 2.** Two map-germs $f$ and $g$ are *equivalent* if $f$ and $g$ are contained in the same $L(n)\times L(p)$-orbit.

**Definition 3.** For a map-germ $f\in\mathcal{E}(n, p)$ we say that $f$ is *k-determined* if for any $g\in\mathcal{E}(n, p)$ which is $k$-jet equivalent to $f$, $f$ and $g$ are equivalent. A map-germ $f$ is called *finitely determined* if there is some positive integer $k$ such that $f$ is $k$-determined.

J. Mather gave the following complete characterization of finitely determined map-germs.

**Theorem [4].** *A map-germ $f\in\mathcal{E}(n, p)$ is finitely determined if* *and only if there is a positive integer k such that*

$$tf(\theta(n))+wf(\theta(p))\supset\mathfrak{M}^k\theta(f)$$

(see [4], [6] for the definitions of $\theta(f)$, $\theta(n)$, $\theta(p)$, $tf:\theta(n)\to\theta(f)$ and $wf:\theta(p)\to\theta(f)$).

However, in general it is difficult to check whether given map-germ $f$ is finitely determined or not (see [2], [3], [5]). On the other hand, from Mather's theorem we easily see that classification of finitely determined singularities of map-germs can be reduced to that of formal mappings. Thus, from now on we consider formal mappings.

Let $\mathfrak{M}$ be the maximal ideal of formal power series algebra $K[[x_1,\cdots,x_n]]$. By $\hat{\mathcal{E}}(n,p)$ we denote the set of all formal mappings $f=(f_1,\cdots,f_p):(K^n,0)\to(K^p,0)$. In the natural way we identify $\hat{\mathcal{E}}(n,p)$ with $\underbrace{\mathfrak{M}\oplus\cdots\oplus\mathfrak{M}}_{p}$ and we regard $\hat{\mathcal{E}}(n,p)$ as $K[[x_1,\cdots,x_n]]$-module. By $\hat{L}(n)$ (resp. $\hat{L}(p)$) we denote the group of formal diffeomorphisms $\varphi:(K^n,0)\to(K^n,0)$ (resp. $\psi:(K^p,0)\to(K^p,0)$). The group $\hat{L}(n)\times\hat{L}(p)$ acts on $\hat{\mathcal{E}}(n,p)$ in the same way as map-germs. And we define also $k$-determinacy and finite determinacy for formal mappings in the same way as for map-germs.

By $H_i$ we denote the vector space spanned by the homogeneous polynomials of degree $i$. By $\mathcal{E}_i(n,p)$ we denote the vector space of homogeneous polynomial mappings of degree $i$, i.e.

$$\mathcal{E}_i(n,p)=\underbrace{H_i\oplus\cdots\oplus H_i}_{p}$$

and $\hat{\mathcal{E}}(n,p)=\mathcal{E}_1(n,p)\oplus\mathcal{E}_2(n,p)\oplus\mathcal{E}_3(n,p)\oplus\cdots$.

**§3. Theorems.** For a formal mapping $f\in\hat{\mathcal{E}}(n,p)$ we represent $f$ as $f=f_{(k)}+f_{(k+1)}+f_{(k+2)}+\cdots$ where

$$f_{(i)}\in\mathcal{E}_i(n,p),\qquad(i=k,\ k+1,\ k+2,\cdots).$$

By $\mathfrak{M}^2\langle\partial f_{(k)}/\partial x\rangle$ we denote the submodule $\mathfrak{M}^2\langle\partial f_{(k)}/\partial x_1,\cdots,\partial f_{(k)}/\partial x_n\rangle$ of $\hat{\mathcal{E}}(n,p)$. We set $B_i=\mathcal{E}_i(n,p)\cap\mathfrak{M}^2\langle\partial f_{(k)}/\partial x\rangle$ and we denote by $G_i$ the compliment linear subspace of $B_i$ in $\mathcal{E}_i(n,p)$, $(i=k+1,k+2,k+3,\cdots)$.

**Theorem A** (Normal form theorem). *Let $f\in\hat{\mathcal{E}}(n,p)$ and $B_i$ and $G_i$ be as above. Then there exists a formal diffeomorphism $\varphi\in L(n)$ such that*

$$f\circ\varphi=f_{(k)}+g_{(k+1)}+g_{(k+2)}+\cdots$$

*where $g_{(i)}\in G_i$, $(i=k+1,\ k+2,\cdots)$.*

**Example 1** (Morse lemma). Suppose that $f\in\hat{\mathcal{E}}(n,1)$ be in the form $f(x_1,\cdots,x_n)=\pm x_1^2\pm\cdots\pm x_n^2+$higher terms. Then $\mathfrak{M}^2\langle\partial f_{(2)}/\partial x\rangle=\mathfrak{M}^2\langle x_1,\cdots,x_n\rangle=\mathfrak{M}^3$. Thus we have $B_i=H_i$ and $G_i=\{0\}$, $(i\geqq 3)$. Therefore the normal form of $f(x_1,\cdots,x_n)$ is $\pm x_1^2\pm\cdots\pm x_n^2$.

**Example 2** (Splitting theorem). Suppose that $f\in\hat{\mathcal{E}}(n,1)$ be in the form $f(x_1,\cdots,x_n)=\pm x_{j+1}^2\pm\cdots\pm x_n^2+$higher terms. Then we have

$\mathfrak{M}^2\langle \partial f_{(2)}/\partial x\rangle=\mathfrak{M}^2\langle x_{j+1}, \cdots, x_n\rangle$. Thus, we can take the vector space spanned by homogeneous polynomials of degree $i$ of variables $x_1, \cdots, x_j$ as $G_i$, $(i\geqq 3)$. Therefore the normal form of $f(x_1, \cdots, x_n)$ is $g(x_1, \cdots, x_j)$ $\pm x_{j+1}^2 \pm \cdots \pm x_n^2$ where order of $g \geqq 3$.

Now, from a variational corollary of above theorem and the classification of 2-jet we obtain the following theorem.

**Theorem B** (Generalized splitting theorem). *For a formal mapping $f\in\hat{\mathcal{E}}(n,p)$ which Jacobian matrix has rank $p-1$, $(n\geqq p)$, there exist formal diffeomorphisms $(\varphi,\psi)\in L(n)\times L(p)$ such that*

(*) $\psi\circ f\circ\varphi=(x_1, \cdots, x_{p-1}, x_1x_p+\cdots+x_ix_{p+i-1}+g(x_{i+1}, \cdots, x_j)+Q_{j+1})$

*where $Q_{j+1}=\pm x_{j+1}^2\pm\cdots\pm x_n^2$ and the second derivatives of $g$ vanish and $i, j$ are uniquely determined by the 2-jet of $f$, $(0\leqq i\leqq p-1$, $p-1\leqq j\leqq n$ and $p+i-1\leqq j)$.*

**Example 3** (Whitney's fold singularity). In the above theorem we consider the simplest case $i=0$ and $j=p-1$. Then the normal form is given by $f=(x_1, \cdots, x_{p-1}, g(x_1, \cdots, x_{p-1})+Q_p)$. By the action of $L(p)$, $f$ is equivalent to $(x_1, \cdots, x_{p-1}, \pm x_p^2\pm\cdots\pm x_n^2)$ i.e. we have Whitney's fold singularity which is 2-determined.

**Example 4** (Whithey's cusp singularity). In the case $n=p=2$ and $i=1$, $j=2$, the normal form is given by $f(x_1, x_2)=(x_1, x_1x_2+g(x_2))$. Then $f(x_1, x_2)$ is finitely determined if and only if $g(x_2)\neq 0$. When the order of $g(x_2)$ equals 3, it is easy to see that $f(x_1, x_2)$ is equivalent to Whitney's cusp singularity $(x_1, x_1x_2+x_2^3)$ which is 3-determined.

**Theorem C.** *Let a formal mapping $f\in\hat{\mathcal{E}}(n,p)$ be in the form (*). We set $\tilde{f}=(x_1, \cdots, x_{p-1}, x_1x_p+\cdots+x_ix_{p+i-1}+g(x_{i+1}, \cdots, x_j)):(\boldsymbol{K}^j, 0)\to(\boldsymbol{K}^p, 0)$. Then $f\in\hat{\mathcal{E}}(n,p)$ is finitely determined if and only if $\tilde{f}\in\hat{\mathcal{E}}(j,p)$ is finitely determined.*

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