120. Representation of the Generator and the Boundary Condition for Semigroups of Operators of Kernel Type

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1. Let $P(t, x, E)$ be a Markov transition probability on a compact manifold D such that $\{P_t, t \geq 0\}$, given by

$$P_t f(x) = \int_D f(y) P(t, x, dy),$$

is a semigroup on $C(D)$. Then, it is known that, under a certain regularity condition, the generator A of $\{P_t, t \geq 0\}$ is represented as a second order integro-differential operator for smooth f in the domain of A([5]). This type of theorems originally go back to Kolmogorov [3], and various versions are obtained as in Yosida [7] and others.

If D is a bounded open domain with smooth boundary in a manifold, and if $\{P_t, t \geq 0\}$ is a diffusion semigroup on $C(\bar{D})$, then smooth functions in the domain of the generator satisfy a boundary condition given by a second order integro-differential operator under a certain regularity condition. This was obtained by Wentzell [6] as a partial extension of Feller [1], [2] for one dimensional diffusion.

Here, in this note, we extend the representation theorems of this type for a complex valued kernel $Q(t, x, E)$. The point is that $Q(t, x, E)$ has not the non-negative property, and the orders of the corresponding integro-differential operators are no more bounded by 2. They depend essentially on the order of $|Q|(t, x, E)$ near the point x as $t \searrow 0$, where $|Q|$ is the measure given by the variation of Q. Neither the semigroup property nor the regularity of

$$Q_t f(x) = \int_D f(y) Q(t, x, dy),$$

as a function of x, are essential for the representations. But, the corresponding propositions for semigroups can be derived easily from Theorems 1–4. The proofs of theorems will be published elsewhere.

2. Let D be a manifold, or an open domain with boundary ∂D in a manifold of dimension N, where the manifold and ∂D are of class C^∞. For a fixed point x in $\bar{D} = D \cup \partial D$,[1] let $\{\xi_k^{(x)}(y), 1 \leq k \leq N\}$ be a local coordinate in a neighbourhood of x, such that $\xi_k^{(x)}(y)$'s are defined and continuous on $\bar{D}$, and $\xi_k^{(x)}(y) = 0$, $1 \leq k \leq N$, if and only if $y = x$. When

1) When D is a manifold, we understand that $\partial D = \phi$ and $\bar{D} = D$.

x is a boundary point of D, we assume

$$\xi_N^{(x)}(y)\geq 0,\ y\in\bar{D},\qquad \xi_N^{(x)}(y)=0, \text{ if and only if } y\in\partial D.$$

Let $Q(t,x,E)$ be a complex valued measure on $(\bar{D},\mathcal{B}_{\bar{D}})$, where $\mathcal{B}_{\bar{D}}$ is the topological σ-field of $\bar{D}$. For $|Q|(t,x,\cdot)$-integrable functions f, let

$$Q_t f(x)=\int_{\bar{D}} f(y)Q(t,x,dy).$$

Let $\mathcal{D}(A_{(x)})$ be the set of all functions f in $C_b(\bar{D})$,[2] such that there is the limit $Af(x)=\lim_{t\searrow 0} 1/t(Q_t f(x)-f(x))$.

Theorem 1. *Let D be bounded, and let x be a point in D and moreover let $Q(t,x,E)$ be a complex valued bounded measure on $(\bar{D},\mathcal{B}_{\bar{D}})$ for each $t>0$. We assume that, for a natural number l,*

(1) $$\int_{\bar{D}}\sum_{k=1}^{N}|\xi_k^{(x)}(y)|^l|Q|(t,x,dy)=O(t),\qquad \text{as } t\searrow 0,$$

(2) $$\xi_\alpha^{(x)}\in\mathcal{D}(A_{(x)}),\text{[3]}\qquad |\alpha|<l.$$

Let n be the smallest of those l such that (1)–(2) *hold. Then,*

(i) *for each f in $\mathcal{D}(A_{(x)})\cap C^n(\bar{D})$,*

(3) $$Af(x)=\sum_{|\alpha|\leq n}a_\alpha(x)D_\alpha f(x) +\int_{\bar{D}\setminus\{x\}}\Big(f(y)-\sum_{|\alpha|<n}1/\alpha!\,D_\alpha f(x)\xi_\alpha^{(x)}(y)\Big)\mu(x,dy),$$

where $a_\alpha(x)$ and $\mu(x,\cdot)$ are independent of f, and $\mu(x,\cdot)$ is a complex valued σ-finite measure on $(\bar{D}\setminus\{x\},\mathcal{B}_{\bar{D}\setminus\{x\}})$ such that

$$\int_{\bar{D}\setminus\{x\}}\sum_{k=1}^{N}|\xi_k^{(x)}(y)|^n|\mu|(x,dy)<\infty.$$

(ii) *If $Q(t,x,E)$ is non-negative, then $n=1$ or 2 and $\mu(x,\cdot)$ is non-negative. When $n=2$, $(a_\alpha(x),|\alpha|=2)$ is non-negative definite.*

(iii) *If, for each neighbourhood U of x,*

$$|Q|(t,x,U^c)=o(t),\qquad \text{as } t\searrow 0,$$

then the measure $\mu(x,\cdot)$ vanishes.

(iv) *If, for some r in $(n-1,n)$,*

$$\int_{\bar{D}}\sum_{k=1}^{N}|\xi_k^{(x)}(y)|^r|Q|(t,x,dy)=O(t),\qquad \text{as } t\searrow 0,$$

then $a_\alpha(x)$ vanishes for $|\alpha|=n$.

Remark 1. (i) If $\mathcal{D}(A_{(x)})$ contains $C^\infty(\bar{D})$, then the representation (3) is unique.

(ii) The conditions (1)–(2) can be replaced by weaker conditions

$$\int_{\bar{D}}\sum_{k=1}^{N}|\xi_k^{(x)}(y)|^l|Q|(t_j,x,dy)=O(t_j),\qquad \text{for a sequence } t_j\searrow 0,$$

$$Q_{t_j}\xi_\alpha^{(x)}(x)-\xi_\alpha^{(x)}(x)=O(t_j),\qquad \text{as } t_j\searrow 0,\ |\alpha|<l.$$

In case D is unbounded, we assume $\lim_{y\to\infty}\sum_{k=1}^{N}|\xi_k^{(x)}(y)|=\infty$, where $y\to\infty$ means that y converges to the point at ∞ in the sense of one-

2) $C^n(\bar{D})(C_b^n(\bar{D}))$ is the set of all n-times continuously differentiable functions (whose derivatives up to order n are bounded) for $0\leq n\leq\infty$.

3) For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_N)$, we write $|\alpha|=\alpha_1+\cdots+\alpha_N$, $\alpha!=\alpha_1!\cdots\alpha_N!$, $\xi_\alpha^{(x)}(y)=(\xi_1^{(x)}(y))^{\alpha_1}\cdots(\xi_N^{(x)}(y))^{\alpha_N}$, and $D_\alpha f(x)=(\partial/\partial\xi_1^{(x)})^{\alpha_1}\cdots(\partial/\partial\xi_N^{(x)})^{\alpha_N}f(x)$.

point compactification of $\bar{D}$. We fix a compact neighbourhood D_0 of x contained in $\bar{D}$, when x is a point of D. If x is a boundary point of D, let D_0 be the intersection of $\bar{D}$ and a compact neighbourhood of x. We write, for abbreviation, $|\xi^{(x)}|^n(y)$ for $\sum_{k=1}^N |\xi_k^{(x)}(y)|^n$.

Theorem 2. *Let D be unbounded, and let x be a point in D and moreover let $Q(t,x,E)$ be a complex valued σ-finite measure on $(\bar{D}, \mathcal{B}_{\bar{D}})$ for $t>0$. We assume, for a natural number l and a positive function $\rho_{(x)}$ in $C_b(\bar{D})$,*

$$(4)\qquad \rho_{(x)}(y)=1, \qquad y \in D_0,$$

$$(5)\qquad \int_{\bar{D}} \rho_{(x)}(y)|\xi^{(x)}|^l(y)|Q|(t,x,dy)=O(t), \qquad \text{as } t\searrow 0,$$

$$(6)\qquad \rho_{(x)}\xi_\alpha^{(x)} \in \mathcal{D}(A_{(x)}), \qquad |\alpha|<l.$$

Let n be the smallest of those l such that (5)–(6) *hold.*

Then, for each f in $\mathcal{D}(A_{(x)})\cap C_b^n(\bar{D})$ such that

$$(7)\qquad f(y)=o(\rho_{(x)}(y)|\xi^{(x)}|^n(y)), \qquad \text{as } y\to\infty,$$

we have

$$Af(x)=\sum_{|\alpha|\le n} a_\alpha(x)D_\alpha f(x) + \int_{\bar{D}\setminus\{x\}} (f(y)-\rho_{(x)}(y)\sum_{|\alpha|<n} 1/\alpha!\, D_\alpha f(x)\xi_\alpha^{(x)}(y))\mu(x,dy),$$

where $\mu(x,\cdot)$ satisfies $\int_{\bar{D}\setminus\{x\}} \rho_{(x)}(y)|\xi^{(x)}|^n(y)|\mu|(x,dy)<\infty$.

The assertions (ii)–(iv) *in Theorem* 1 *hold true, where $\bar{D}$ in* (iv) *and $|Q|(t,x,U^c)$ in* (iii) *are replaced by D_0 and $\int_{U^c}\rho_{(x)}(y)|Q|(t,x,dy)$, respectively.*

For Theorem 2, similar assertions as in Remark 1 hold true.

3. Boundary conditions. Let x be a boundary point of D. For a pair of natural numbers (n,n') such that $n'\le n$, we write

$I_{(n,n')}=\{\alpha\,|\,(|\alpha|-\alpha_N)/n+\alpha_N/n'\le 1\}$, $I^0_{(n,n')}=\{\alpha\,|\,(|\alpha|-\alpha_N)/n+\alpha_N/n'<1\}$.

We also write $\langle\xi^{(x)}\rangle^{(n,n')}(y)$ for $\sum_{k=1}^{N-1}|\xi_k^{(x)}(y)|^n+(\xi_N^{(x)}(y))^{n'}$.

Theorem 3. *Let D be bounded, and let x be a boundary point and moreover let $Q(t,x,E)$ be a complex valued bounded measure on $(\bar{D}, \mathcal{B}_{\bar{D}})$ for $t>0$. We assume that, for a pair (n,n') with $n'\le n$, there is a function f_0 in $C^n(\bar{D})$ such that*

$$Q_t f_0(x)-f_0(x)\neq o\left(\int_{\bar{D}}\langle\xi^{(x)}\rangle^{(n,n')}(y)|Q|(t,x,dy)\right), \qquad \text{as } t\searrow 0.$$

Then, each f in $\mathcal{D}(A_{(x)})\cap C^n(\bar{D})$ satisfies a non-trivial boundary condition

$$Lf(x)=\delta(x)Af(x)+\sum_{\alpha\in I_{(n,n')}} b_\alpha(x)D_\alpha f(x) + \int_{\bar{D}\setminus\{x\}}(f(y)-\sum_{\alpha\in I^0_{(n,n')}} 1/\alpha!\, D_\alpha f(x)\xi_\alpha^{(x)}(y))\nu(x,dy)=0,$$

where $\nu(x,\cdot)$ is a complex valued σ-finite measure on $(\bar{D}\setminus\{x\}, \mathcal{B}_{\bar{D}\setminus\{x\}})$ such that $\int_{\bar{D}\setminus\{x\}}\langle\xi^{(x)}\rangle^{(n,n')}(y)|\nu|(x,dy)<\infty$.

If $Q(t, x, \cdot)$ is non-negative, then the above assumption is satisfied for $(n, n')=(2,1)$. In this case, $\delta(x)\leq 0$, $\{b_\alpha(x), |\alpha|=2, \alpha_N=0\}$ is non-negative definite, $b_{(0,\dots,0,1)}(x)\geq 0$ and $\nu(x, \cdot)$ is non-negative.

Theorem 4. *Let D be unbounded, let x be a boundary point and let $Q(t, x, E)$ be a complex valued σ-finite measure on $(\bar{D}, \mathcal{B}_{\bar{D}})$ for $t>0$. We assume that there are a positive function $\rho_{(x)}$ in $C_b(\bar{D})$, which satisfies (4), and a function f_0 in $C_b^n(\bar{D})$ such that, for a pair (n, n'),*

$$Q_t f_0(x)-f_0(x)$$

$$\neq o\left(\left\{\int_{D_0}\langle\xi^{(x)}\rangle^{(n,n')}(y)+\int_{\bar{D}_0\setminus D_0}\rho_{(x)}(y)|\xi^{(x)}|^n(y)\right\}|Q|(t, x, dy)\right), \quad \text{as } t\searrow 0.$$

Then, each f in $\mathcal{D}(A_{(x)})\cap C_b^n(\bar{D})$ such that (7) holds satisfies a non-trivial boundary condition

$$Lf(x)=\delta(x)Af(x)+\textstyle\sum_{\alpha\in I_{(n,n')}} b_\alpha(x)D_\alpha f(x)$$

$$+\int_{\bar{D}\setminus\{x\}}(f(y)-\rho_{(x)}(y)\sum_{\alpha\in I^0_{(n,n')}} 1/\alpha!\, D_\alpha f(x)\xi_\alpha^{(x)}(y))\nu(x, dy)=0,$$

where $\nu(x, \cdot)$ satisfies

$$\left(\int_{D_0\setminus\{x\}}\langle\xi^{(x)}\rangle^{(n,n')}(y)+\int_{\bar{D}\setminus D_0}\rho_{(x)}(y)|\xi^{(x)}|^n(y)\right)|\nu|(x, dy)<\infty.$$

If $Q(t, x, E)$ is non-negative and, for some $n\geq 2$ and $n'\geq 1$,

$$\int_{D_0}\langle\xi^{(x)}\rangle^{(n,n')}(y)|Q|(t, x, dy)$$

$$\neq o\left(\int_{\bar{D}\setminus D_0}\rho_{(x)}(y)|\xi^{(x)}|^n(y)|Q|(t, x, dy)\right), \qquad \text{as } t\searrow 0,$$

then the above assumption is satisfied for $(n, n')=(2,1)$. In this case, $\delta(x)\leq 0$, $(b_\alpha(x), |\alpha|=2, \alpha_N=0)$ is non-negative definite, $b_{(0,\dots,0,1)}(x)\geq 0$, and $\nu(x, \cdot)$ is non-negative.

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