114. A Note on the Approximate Functional Equation for $\zeta^2(s)$

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1. Let $0\leq\sigma\leq 1$, $t\geq 2$, $XY=t^2/4\pi^2$, $xy=t/2\pi$, and put

$$\zeta(s)=\sum_{n\leq x} n^{-s}+\chi(s)\sum_{n\leq y} n^{s-1}+E(s,x), \tag{1}$$
$$\zeta^2(s)=\sum_{n\leq X} d(n)n^{-s}+\chi^2(s)\sum_{n\leq Y} d(n)n^{s-1}+D(s,X)$$

where d is the divisor function and $\chi(s)=2^s\pi^{s-1}\sin(s\pi/2)\Gamma(1-s)$. Hardy and Littlewood [1] proved that

$$E(s,x)\ll x^{-\sigma}+t^{1/2-\sigma}y^{\sigma-1} \tag{2}$$

as well as

$$D(s,X)\ll X^{1/2-\sigma}\left(\frac{X+Y}{t}\right)^{1/4}\log t.$$

Later Titchmarsh [5] replaced the latter by

$$D(s,X)\ll (X+Y)^{1/2-\sigma}\log t. \tag{3}$$

Also we should note that Jutila [4] remarked recently that (3) is a consequence of Voronoi's summation formula.

The arguments of these authors are rather elaborated, mainly because they treated $\zeta^2(s)$ directly, i.e. without recoursing to the known approximations for $\zeta(s)$. Here we shall show that if we make use of Dirichlet's device:

$$\sum_{n\leq N} d(n)a_n=2\sum_{n\leq\sqrt{N}}\sum_{m\leq N/n} a_{nm}-\sum_{n\leq\sqrt{N}}\sum_{m\leq\sqrt{N}} a_{nm}, \tag{4}$$

then, as far as the most interesting case $X=Y=t/2\pi$ is concerned, (3) is a quite simple consequence of (2).

For this end let $U=t/2\pi$, $u=\sqrt{U}$. Then by (4) we have

$$\sum_{n\leq U} d(n)n^{-s}+\chi^2(s)\sum_{n\leq U} d(n)n^{s-1}$$
$$=2\sum_{m\leq u} m^{-s}\sum_{n\leq U/m} n^{-s}+2\chi^2(s)\sum_{m\leq u} m^{s-1}\sum_{n\leq U/m} n^{s-1}$$
$$-\Big(\sum_{m\leq u} m^{-s}\Big)^2-\chi^2(s)\Big(\sum_{m\leq u} m^{s-1}\Big)^2.$$

And by (1) this is equal to

$$2\sum_{m\leq u} m^{-s}\Big\{\zeta(s)-\chi(s)\sum_{n\leq m} n^{s-1}-E(s,U/m)\Big\}$$
$$+2\chi^2(s)\sum_{m\leq u} m^{s-1}\Big\{\zeta(1-s)-\chi(1-s)\sum_{n\leq m} n^{-s}-E(1-s,U/m)\Big\}$$
$$+2\chi(s)\sum_{m\leq u} m^{-s}\sum_{n\leq u} n^{s-1}-(\zeta(s)-E(s,u))^2.$$

Then, after some rearrangement, we get

$$\zeta^2(s)=\sum_{n\le U} d(n)n^{-s}+\chi^2(s)\sum_{n\le U} d(n)n^{s-1}+2\chi(s)\sum_{n\le u}\frac{1}{n} \tag{5}$$
$$+2\sum_{m\le u} m^{-s}E(s, U/m)+2\chi^2(s)\sum_{m\le u} m^{s-1}E(1-s, U/m)+E^2(s, u).$$

Inserting (2) into this we readily obtain (3) for the case $X=Y=t/2\pi$.

2. Next, let us see what the above argument will yield if we replace (2) by a more precise estimate obtainable by the method of Riemann and Siegel (see e.g. [6, Chap. 4]). This gives, for $1\le x \le t/2\pi$,

$$E(s, x)=(2\pi)^{s-2}e^{-\pi is/2}\Gamma(1-s)y^{s-1}\int_L \exp\left(\frac{xi}{4\pi y}(w-2\pi i\{y\})^2\right.$$
$$\left.+x(w-2\pi i\{y\})-[x]w\right)(e^w-1)dw+O((t^{1/2-\sigma}y^{\sigma-1}+x^{-\sigma})t^{-1/6}),$$

where $\{y\}=y-[y]$, and L is a straight line in the direction $\arg w=\pi/4$ passing between 0 and $2\pi i$. Thus we have, in (5),

$$\sum_{m\le u} m^{-s}E(s, U/m)$$
$$=(2\pi)^{s-2}e^{-\pi is/2}\Gamma(1-s)\sum_{m\le u}\frac{1}{m}\int_L \exp\left(\frac{ti}{8\pi^2m^2}w^2+\left\{\frac{t}{2\pi m}\right\}w\right)$$
$$\times(e^w-1)^{-1}dw+O(t^{1/3-\sigma}\log t).$$

Then we deform L to the curve composed of two parts: $w=\{\lambda e^{\pi i/4}, \lambda\ge\delta$ and $\lambda\le-\delta\}$ and $\{w=\delta e^{i(\theta+\pi/4)}, 0\le\theta\le\pi\}$ where $\delta>0$, and let δ tend to 0. This gives

$$\int_L=-\pi i+\varepsilon\int_0^\infty e^{-(t/8\pi^2m^2)\lambda^2}\frac{\sin((\{t/2\pi m\}-1/2)\lambda/\varepsilon)}{\sin\lambda/2\varepsilon}d\lambda,$$

where $\varepsilon=e^{\pi i/4}$.

Hence we have

$$\sum_{m\le u} m^{-s}E(s, U/m)=-\frac{\chi(s)}{2}\sum_{m\le u}\frac{1}{m}$$
$$+\varepsilon(2\pi)^{s-2}e^{-\pi is/2}\Gamma(1-s)\int_0^\infty\sum_{m\le u}\frac{e^{-(t/8\pi^2m^2)\lambda^2}}{m}\frac{\sin((\{t/2\pi m\}-1/2)\lambda/\varepsilon)}{\sin\lambda/2\varepsilon}d\lambda \tag{6}$$
$$+O(t^{1/3-\sigma}\log t).$$

To estimate this integrand, we introduce the Fourier expansion: For $0<v<1$

$$\frac{\sin((v-1/2)\lambda/\varepsilon)}{\sin\lambda/2\varepsilon}=-8\pi\sum_{j=1}^\infty\frac{j}{4\pi^2j^2+\lambda^2 i}\sin(2\pi jv),$$

which is boundedly convergent uniformly for all real λ. Then, invoking an idea of Hooley [3, p. 104], we have, for any $J\ge1$,

$$\frac{\sin((\{t/2\pi m\}-1/2)\lambda/\varepsilon)}{\sin\lambda/2\varepsilon}=-8\pi\sum_{j=1}^J\frac{j}{4\pi^2j^2+\lambda^2i}\sin\left(\frac{jt}{m}\right) \tag{7}$$
$$+O\left(\sum_{j=0}^\infty A_j(J)\cos\left(\frac{jt}{m}\right)\right)+O\left(I\left(\frac{t}{2\pi m}\right)\right),$$

where

$$A_j(J)=O\left(\min\left(\frac{\log J}{J}, \frac{J}{j^2}\right)\right),$$

and I is the characteristic function of the set of integers; here the O-constants are all independent of λ.

Now, we divide the integrand in question into two parts according to $m\le t^{1/3}$ and $t^{1/3}<m\le u$. Since the sin-factor is $O(1)$ we see readily that the first part contributes to (6) the amount of $O(t^{1/3-\sigma})$. As for the second part we consider, instead, the estimation of the sum

$$(8)\qquad \sum_{M<m\le 2M} \frac{\sin((\{t/2\pi m\}-1/2)\lambda/\varepsilon)}{\sin \lambda/2\varepsilon}, \qquad (t^{1/3}\ll M\ll t^{1/2}).$$

But we have (7), so the problem is reduced to that of

$$\sum_{M<m\le 2M} e^{jtt/m}.$$

Appealing to van der Corput's method (see e.g. [6, p. 90]), this is estimated to be $O(M^{-1/2}t^{1/2}j^{1/2})$ if $j\neq 0$. Hence setting $J=t^{-1/3}M(\log t)^{1/2}$ in (7) we find that (8) is $O(t^{1/3}(\log t)^{1/2})$ uniformly for all real λ. Then, summing partially over m and integrating with respect to λ we conclude that the second part in question of the integrand of (6) contributes to it the amount of $O(t^{1/3-\sigma}(\log t)^{3/2})$.

Hence we have found

$$\sum_{m\le u} m^{-s}E(s, U/m) = -\frac{\chi(s)}{2}\sum_{m\le u}\frac{1}{m}+O(t^{1/3-\sigma}(\log t)^{3/2}).$$

Inserting this into (5) we obtain the following improvement on (3):

Theorem.

$$(9)\qquad D(s, t/2\pi)=O(t^{1/3-\sigma}(\log t)^{3/2}).$$

Remark. By elaborating our argument one may probably replace (9) by an asymptotic expansion, i.e. an analogue for $\zeta^2(s)$ of the Riemann-Siegel formula for $\zeta(s)$. Also one may treat the non-symmetric case (i.e. $X\neq Y$) as well. Further we should remark that our theorem may be incorporated into the asymptotic evaluation of the fourth power moment of $\zeta(s)$ (cf. [2]). To these and further improvements we shall return elsewhere.

Added in proof. (i) A refinement of the above argument yields

$$(10)\qquad D\left(s, \frac{t}{2\pi}\right)=-2\left(\frac{\pi}{t}\right)^{1/2}\Delta\left(\frac{t}{2\pi}\right)\chi(s)+O(t^{1/4-\sigma}),$$

in which $\Delta(x)$ is defined as (12.1.2) of [6]. Obviously this is better than (9), and gives an Ω-result for $D(s, t/2\pi)$.

(ii) Professor Jutila kindly sent us a preprint in which he proved a result similar to (but weaker than) (9). Also, in a letter to us, he indicated that his argument might yield a result like (10).

References

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