110. A Note on Γ-Rings

By Bhavanari SATYANARAYANA
Department of Mathematics, Nagarjuna University

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Introduction. Throughout the paper, M stands for a Γ-ring, as defined by Barnes [1]. We shall utilize the standard notations and definitions in Barnes [1] and Hsu [2]. In [2] Hsu has introduced the notion of g-prime ideals in Γ-rings and proved that for any ideal A of the Γ-ring M, the radical $r_g(A)$ of A (that is, the set of all elements x of M such that every g-system containing x contains an element of A) is the intersection of all g-prime ideals containing A. In this paper we introduce the notion of g-halfprime ideals in Γ-rings and prove that an ideal A of the Γ-ring M is g-halfprime if and only if $A=r_g(A)$.

Preliminary definitions. If a is an element of the Γ-ring M, then $\langle a\rangle$ denotes the principal ideal generated by a. If S is a subset of M, we call S an *sp-system* if $S=\emptyset$ or $a\in S$ implies $\langle a\rangle^2\cap S\neq\emptyset$. A non-empty subset S of M is called a *g-sp-system* if S contains an sp-system S' such that $g(x)\cap S'\neq\emptyset$ for every element x of S, where S' is called a *kernel* of S. An ideal I of M is said to be *g-halfprime* if $C(I)=M\backslash I$ is a g-sp-system.

Example. Consider Z, the ring of integers, as a Γ-ring with $\Gamma=Z$. Let p, q be two distinct prime numbers. Define $g(a)=\langle\{a, pq\}\rangle$. Now $g(pq)=\langle pq\rangle$ and hence $C(\langle pq\rangle)$ is g-sp-system with kernel $C(\langle pq\rangle)$, which is not a g-system.

Suppose K is a subset of M and satisfies the condition: For each $a\in K$, there exists an sp-system $S\subseteq K$ such that $g(a)\cap S\neq\emptyset$. Then consider the set X, which is the union of all sp-systems which are contained in K. One can easily verify that K is a g-sp-system with kernel X. Hence a subset K of M is a g-sp-system if and only if K satisfies the condition: For each $a\in K$, there exists an sp-system $S\subseteq K$, such that $g(a)\cap S\neq\emptyset$.

Main Theorem. Before proving our main theorem, we prove the following

Lemma. *If S is an sp-system and $x\in S$, then there exists an m-system X* (Def. 3.2. in [2]) *such that $x\in X$ and $X\subseteq S$.*

Poof. Let S be an sp-system and x an element of S. Then there exists an element $x_1\in\langle x\rangle^2\cap S$. Again since S is an sp-system, there exists $x_2\in\langle x_1\rangle^2\cap S$. If we continue this process, we get a sequence $\{x_i\}$ of elements in S with $x_0=x$ and $x_{i+1}\in\langle x_i\rangle^2\cap S$ for $i\geq 0$. Now x_i

$\in \langle x_{i-1}\rangle^2$, $\langle x_{i-1}\rangle^2 \subseteq \langle x_{i-1}\rangle$ for each i, so that $\langle x_0\rangle \supseteq \langle x_1\rangle \supseteq \langle x_2\rangle \supseteq \langle x_3\rangle \supseteq \cdots$. It is easy to verify that $x_j \in \langle x_i\rangle\langle x_j\rangle \cap X$ for $i \leq j$. Hence $X = \{x_0, x_1, x_2, \cdots\}$ is an m-system such that $x = x_0 \in X$ and $X \subseteq S$. Hence the lemma.

Theorem. *Let M be a Γ-ring and A be an ideal of M. Then A is g-halfprime if and only if $A = r_g(A)$.*

Proof. Suppose A is g-halfprime. Clearly $A \subseteq r_g(A)$. To show $r_g(A) \subseteq A$, let $a \in rg(A)$. Suppose $a \notin A$. Since $C(A)$ is a g-sp-system there exists an element x and an sp-system K such that $x \in g(a)$, $x \in K$ and $K \subseteq C(A)$. Now by above Lemma, there exists an m-system K^* such that $x \in K^*$ and $K^* \subseteq K$. Write $Q = \{y \in C(A) \mid g(y) \cap K^* \neq \emptyset\}$. Clearly $K^* \subseteq Q \subseteq C(A)$, $a \in Q$ and Q is a g-system with kernel K^*. By Zorn's Lemma (applied to the class of all ideals I, such that $I \cap Q = \emptyset$, $I \supseteq A$), there exists an ideal P maximal with respect to the properties $P \cap Q = \emptyset$ and $P \supseteq A$. By the proof of Theorem 3.8. in [2], if follows that P is g-prime. Since $a \notin P$ and P is g-prime containing A, Theorem 3.8. in [2] shows that $a \notin r_g(A)$. This is a contradiction. Hence $A = r_g(A)$.

Conversely, suppose that $A = r_g(A)$. To show that A is g-halfprime we have to show that $C(A)$ is a g-sp-system. If $r(A)$ is the intersection of all prime ideals containing A, then $C(r(A))$ is an sp-system. Now we show that $C(A)$ is a g-sp-system with kernel $C(r(A))$. To show this let $x \in C(A)$. Now $x \notin A = r_g(A)$ and so there is a g-system Y such that $x \in Y$ and $Y \cap A = \emptyset$. Let X be any kernel of Y. Then since $x \in Y$ and X is a kernel of Y there exists an element $z \in g(x) \cap X$. Now $z \in X$, X is an m-system and $X \cap A = \emptyset$. By Theorem 7 of [1], $z \notin r(A)$. $z \notin r(A)$ and $z \in g(x)$ imply $g(x) \cap C(r(A)) \neq \emptyset$. Hence the theorem.

By the property (2) in Theorem 4.3 [2], we have

Corollary. *Intersection of any collection of g-prime ideals is a g-halfprime ideal.*

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References

[1] W. E. Barnes: On the Γ-rings of Nobusawa. Pacific J. Math., **18**, 411–422 (1966).

[2] D. F. Hsu: On prime ideals and primary decompositions in Γ-rings. Math. Japonicae, **21**, 455–460 (1976).