12. *Hodge Filtrations on Gauss-Manin Systems. II*

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Let $f: X \to Y$ be a projective morphism of algebraic manifolds. The theory of Deligne, Gabber, Beilinson, and Bernstein describes the decomposition of the direct image $\boldsymbol{R}f_*\boldsymbol{C}_X$ in $D_c^b(\boldsymbol{C}_Y)$, and gives the Poincaré duality and the hard Lefschetz theorem (cf. [3]). We prove the theorem for a one-parameter projective family (i.e., f is flat projective and $\dim Y=1$) without assuming algebraicity (cf. Theorem (1.1) and Corollary (1.2)). We use essentially the theory of filtered $\mathcal{D}$-Modules [1], [5], which enables us to apply the theory of limit mixed Hodge structure of Steenbrink.

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§ 1. Let $f: Y \to S$ be a projective morphism of complex manifolds with $\dim Y=n+1$ and $\dim S=1$. In [5], we defined the Gauss-Manin system $\int_f \mathcal{O}_Y$ in $DF(\mathcal{D}_S)$, such that $DR_S\left(\int_f \mathcal{O}_Y\right) \simeq \boldsymbol{R}f_*\boldsymbol{C}_Y$ in $D_c^b(\boldsymbol{C}_S)$ (cf. [1], [4]).

(1.1) **Theorem.** 1) *We have the isomorphisms*

$$\int_f \mathcal{O}_Y \simeq \sum_k \int_f^k \mathcal{O}_Y[-k] \qquad \text{in } DF(\mathcal{D}_S), \tag{1.1.1}$$

and

$$\int_f^k \mathcal{O}_Y \simeq \mathcal{H}_\Sigma^0\left(\int_f^k \mathcal{O}_Y\right) \oplus {}^\pi\left(\int_f^k \mathcal{O}_Y|_{S^*}\right) \qquad \text{for any } k \in \boldsymbol{Z}, \tag{1.1.2}$$

as filtered $\mathcal{D}_S$-Modules. Here,

$$\int_f^k \mathcal{O}_Y = \mathcal{H}^k\left(\int_f \mathcal{O}_Y\right),$$

$\Sigma = S - S^*$ is the set of the critical values of f and ${}^\pi\mathcal{M}$ is the minimal extension of a regular holonomic system $\mathcal{M}$ on S^* (i.e., $\mathcal{H}_\Sigma^0({}^\pi\mathcal{M}) = \mathcal{H}_\Sigma^0({}^\pi\mathcal{M}^*) = 0$ *and* ${}^\pi\mathcal{M}|_{S^*} \simeq \mathcal{M}$).

2) *Let L be a relatively ample line bundle on Y and let us also denote by L the operator defined by the cup product of $c_1(L)$. Then we have the isomorphisms for any $k \in \boldsymbol{Z}_+$*

$$L^k: \int_f^{n-k} \mathcal{O}_Y \overset{\sim}{\to} \int_f^{n+k} \mathcal{O}_Y\{k\}, \tag{1.1.3}$$

and

$$\int_f^{n-k} \mathcal{O}_Y \simeq \left(\int_f^{n+k} \mathcal{O}_Y\right)^* \{-n\}, \tag{1.1.4}$$

which are compatible with the decomposition (1.1.2).

In particular, $\int_f^k \mathcal{O}_Y$ *is Cohen-Macaulay and self dual of weight* k, [5].

Remarks. 1. It is conjectured in [1] that the decompositions (1.1.1) and (1.1.2) hold in a more general case.

2. The Hodge filtration $\mathcal{F}^\cdot$ on ${}^\pi\left(\int_f^k \mathcal{O}_Y|_{S^*}\right)$ is determined by the filtration on S^*: i.e., $\mathcal{F}^p\, {}^\pi\left(\int_f^k \mathcal{O}_Y|_{S^*}\right)=\sum_{i\geq 0}\partial_t^i \hat{\mathcal{F}}^{p+i}(\mathcal{L}^k)$ (cf. [5], [1, §2]).

(1.2) **Corollary.** 1) *We have the isomorphisms*

$$Rf_*C_Y \simeq \sum_k R^kf_*C_Y[-k] \qquad \text{in } D_c^b(C_Y),$$

and

$$R^kf_*C_Y \simeq \mathcal{H}_\Sigma^0(R^kf_*C_Y)\oplus j_*(R^kf_*C_Y|_{S^*}) \qquad \text{for any } k\in \mathbf{Z},$$

where $j: S^* \hookrightarrow S$ *is the natural inclusion.*

2) *We have the isomorphisms for any* $k \in \mathbf{Z}_+$

$$L^k: j_*(R^{n-k}f_*C_Y|_{S^*}) \xrightarrow{\sim} j_*(R^{n+k}f_*C_Y|_{S^*})$$

$$j_*(R^{n-k}f_*C_Y|_{S^*}) \simeq \mathcal{H}om_C(j_*(R^{n+k}f_*C_Y|_{S^*}), C_S)$$

$$L^k: \mathcal{H}_\Sigma^0(R^{n+1-k}f_*C_Y) \xrightarrow{\sim} \mathcal{H}_\Sigma^0(R^{n+1+k}f_*C_Y)$$

$$\mathcal{H}_\Sigma^0(R^{n+1-k}f_*C_Y)|_\Sigma \simeq \mathcal{H}om_C(\mathcal{H}_\Sigma^0(R^{n+1+k}f_*C_Y)|_\Sigma, C_\Sigma).$$

Remark. If $\dim S>1$, the decomposition $Rf_*C_Y\simeq\sum R^kf_*C_Y[-k]$ does not hold in general. We need the complex of intersection cohomology sheaf of Deligne-Goresky-MacPherson [3].

§2. **Vanishing cohomology sheaves.** Let $f: Y\to S$ be a one-parameter projective family on a unit disc. We assume that Y is smooth and that $Y_0=f^{-1}(0)$ is a divisor with normal crossings whose irreducible components are nonsingular. We set

$$m := LCM\{\text{mult}_{E_i} f^*t\},$$

where $Y_0=\bigcup E_i$ is the decomposition into irreducible components and t is a local coordinate on S.

Let U be a universal covering of $S^*=S-\{0\}$. We set $Y_\infty = Y\times_S U$ and $\vec{j}: Y_\infty\to Y$ a natural morphism. Following Deligne (cf. SGA7, XIV), we define the complexes of sheaves $R\Psi C$ and $R\Phi C$ on Y_0 by

$$R\Psi C := R\vec{j}_*C_{Y_\infty}|_{Y_0}$$

and

$$R\Phi C := \text{Coker}\,(C_{Y_0}\to R\Psi C).$$

Let $(R\Psi C)_\alpha$ (resp. $(R\Phi C)_\alpha$) be the subcomplex of $R\Psi C$ (resp. $R\Phi C$), on which M_s acts as the scalar multiplication $\alpha\,\text{id}$, where M_s is the semisimple part of the monodromy M and α is a complex number (cf. [7, (2.13)]). By the monodromy theorem, we have $R\Psi C\simeq\oplus_i(R\Psi C)_{e(i/m)}$ and $R\Phi C\simeq\oplus_i(R\Phi C)_{e(i/m)}$, where $e(i/m) := \exp(2\pi\sqrt{-1}\,i/m)$.

Let $\Omega_Y^\cdot$ be the complex of holomorphic differential forms. We set

$$\mathcal{A}^p := \{w\in\Omega_Y^p : df\wedge w=0\}$$

and

$$\mathcal{B}^p := \mathcal{A}^p / df_\wedge \Omega_Y^{p-1} \qquad \text{for } p \in \boldsymbol{Z},$$

where $df := f^* dt$. $\mathcal{A}^\cdot$ and $\mathcal{B}^\cdot$ are complexes of sheaves on Y with the differentiation d.

(2.1) **Proposition.** 1) *There are finite decreasing filtrations $V^\cdot$ on $\mathcal{A}^\cdot / t\mathcal{A}^\cdot$ and on $\mathcal{B}^\cdot$ such that we have quasi-isomorphisms*

$$\mathrm{Gr}_V^i (\mathcal{A}^\cdot / t\mathcal{A}^\cdot)[1]|_{Y_0} \simeq (\boldsymbol{R}\Psi\boldsymbol{C})_{e(-i/m)}$$

and

$$\mathrm{Gr}_V^i (\mathcal{B}^\cdot[1])|_{Y_0} \simeq (\boldsymbol{R}\Phi\boldsymbol{C})_{e(-i/m)} \qquad \textit{for } i=1, \cdots, m.$$

We have $\mathrm{Gr}_V^i = 0$ *for* $i \leq 0$ *or* $i > m$.

Moreover, they induce strictly compatible filtrations $V^\cdot$ on $\boldsymbol{R}\Gamma(Y_0, (\mathcal{A}^\cdot / t\mathcal{A}^\cdot)[1])$ and on $\boldsymbol{R}\Gamma(Y_0, \mathcal{B}^\cdot[1])$ respectively.

2) *The stupid filtration $\{\sigma_{\geq p}\}$ on $(\mathcal{A}^\cdot / t\mathcal{A}^\cdot)[1]$ and on $\mathcal{B}^\cdot[1]$ induces strictly compatible filtrations $F^\cdot$ on $\boldsymbol{R}\Gamma(Y_0, \mathrm{Gr}_V^\cdot (\mathcal{A}^\cdot / t\mathcal{A}^\cdot)[1])$ and on $\boldsymbol{R}\Gamma(Y_0, \mathrm{Gr}_V^\cdot \mathcal{B}^\cdot[1])$ respectively. They coincide with the Hodge filtrations of the mixed Hodge structures of Steenbrink on $H^\cdot(Y_0, \boldsymbol{R}\Psi\boldsymbol{C}) \simeq H^\cdot(Y_\infty, \boldsymbol{C})$ and on $H^\cdot(Y_0, \boldsymbol{R}\Phi\boldsymbol{C})$, and* Coker $(H^k(Y_0, \boldsymbol{R}\Psi\boldsymbol{C}) \to H^k(Y_0, \boldsymbol{R}\Phi\boldsymbol{C}))$ *has a pure Hodge structure of weight $k+1$.*

3) *We have the filtered isomorphisms for any $k \in \boldsymbol{Z}_+$*

$$L^k : H^{n-k}(Y_0, \mathrm{Gr}_V^\cdot (\mathcal{A}^\cdot / t\mathcal{A}^\cdot)[1]) \xrightarrow{\sim} H^{n+k}(Y_0, \mathrm{Gr}_V^\cdot (\mathcal{A}^\cdot / t\mathcal{A}^\cdot)[1])\{k\}$$

$$L^k : H^{n-k}(Y_0, \mathrm{Gr}_V^\cdot (\mathcal{B}^\cdot[1])) \xrightarrow{\sim} H^{n+k}(Y_0, \mathrm{Gr}_V^\cdot (\mathcal{B}^\cdot[1]))\{k\}.$$

This proposition follows from the results of Steenbrink [6], [7]. The filtration $V^\cdot$ is induced by the $\tilde{t}$-adic filtration on $\tilde{\mathcal{A}}^\cdot$, cf. [5].

§ 3. **Proof of the theorem.** It is sufficient to prove the decomposition (1.1.2), the hard Lefschetz Theorem (1.1.3) and the strict compatibility of the Hodge filtration on $\int_f \mathcal{O}_Y$. In fact, (1.1.1) is reduced to the last two statements by an argument of Deligne [2], and (1.1.4) follows from [5, (1.4)] and (1.1.1).

Thus we may assume that f is flat, S is a unit disc, and $\sum = \{0\}$. $\int_f \mathcal{O}_Y$ is calculated as the direct image of the complex of sheaves on Y

$$\mathcal{C}^\cdot = \Omega_Y^\cdot[\partial_t][1]$$

with the differentiation $d - \partial_t df_\wedge$ and the Hodge filtration

$$\mathcal{F}^p(\mathcal{C}^\cdot) = \textstyle\sum_{i \leq -p-1} \Omega_Y^\cdot \partial_t^i[1].$$

Using the arguments similar to those in [5, (2.5)] and [7, (1.13)], we may assume that Y_0 is a divisor with normal crossings as in § 2. We define the increasing filtration $U_.$ on $\mathcal{C}^\cdot$ by

$$U_k(\mathcal{C}^\cdot) := (\textstyle\sum_{i<k} \Omega_Y^\cdot \partial_t^i + \mathcal{A}^\cdot \partial_t^k)[1] \qquad \text{for } k \in \boldsymbol{Z}.$$

We have the filtered isomorphisms

$$\mathrm{Gr}_0^U \mathcal{C}^\cdot \simeq \mathcal{A}^\cdot[1]$$

and

$$\mathrm{Gr}_k^U \mathcal{C}^\cdot \simeq \mathcal{B}^\cdot[1](k) \qquad \text{for } k \geq 1.$$

The next proposition gives the desired result combined with

Proposition (2.1), since (1.1.2) follows from the local classification of regular holonomic systems on S, the local invariant cycle theorem [6] and Propositions (2.1) 2) and (3.1).

(3.1) **Proposition.** *The induced filtrations $U_{\bullet}$ and $\mathcal{F}^{\cdot}$ on $\boldsymbol{R}f_{*}\mathcal{C}^{\cdot}$ are strictly compatible.*

Set $\bar{\mathcal{C}}^{\cdot} := \mathcal{C}^{\cdot}/U_0\mathcal{C}^{\cdot}$, so that

$$0 \longrightarrow \boldsymbol{R}^k f_{*}\mathcal{A}^{\cdot}[1] \longrightarrow \boldsymbol{R}^k f_{*}\mathcal{C}^{\cdot} \longrightarrow \boldsymbol{R}^k f_{*}\bar{\mathcal{C}}^{\cdot} \longrightarrow 0$$

is exact [5]. Using the theory of microlocalization, we can show that $U_{\bullet}$ is strict on $\boldsymbol{R}f_{*}\bar{\mathcal{C}}^{\cdot}$, and hence on $\boldsymbol{R}f_{*}\mathcal{C}^{\cdot}$. Then the strict compatibility of $\mathcal{F}^{\cdot}$ follows from Proposition (2.1).

Remark. $U_i\left(\int_f^k \mathcal{O}_Y\right)$ is the $\mathcal{O}_S$-subModule generated by $w \in \int_f^k \mathcal{O}_Y$ such that $(t\partial_t - \alpha)^m w = 0$ for some $m \in N$ and $\alpha > -1-i$.

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