105. Boundedness of Singular Integral Operators of Calderón Type

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§1. **Introduction.** Let $K(x,y)$ be a kernel satisfying $|K(x,y)| \leqq \text{Const.}/|x-y|$ for any pair (x,y) of real numbers with $x\neq y$. We say that $K(x,y)$ is of type 2 if $Kf(x)=\lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|<1/\varepsilon} K(x,y)f(y)\,dy$ exists almost everywhere for any $f\in L^2$ and $\|K\|_2=\sup\{\|Kf\|_2/\|f\|_2;\ f\in L^2\}<\infty$, where L^2 denotes the space of square integrable functions $f(x)$ on the real line with norm $\|f\|_2=\left\{\int_{-\infty}^{\infty}|f(x)|^2dx\right\}^{1/2}$. For the harmonic analysis on curves, A. Calderón investigated kernels $C[\phi](x,y)=1/\{(x-y)+i(\phi(x)-\phi(y))\}$ for real-valued functions $\phi(x)$ and, in [2], he showed that $C[\phi]$ is of type 2 as long as $\|\phi'\|_\infty=\text{ess.}\sup_x|\phi'(x)|$ is sufficiently small. Using this theorem he also studied kernels

$$(1)\qquad C[h,\phi](x,y)=\frac{1}{x-y}h\left\{\frac{\phi(x)-\phi(y)}{x-y}\right\}$$

for complex-valued functions $h(t)$ and real-valued functions $\phi(x)$. In [5], R. Coifman-A. McIntosh-Y. Meyer showed that $C[\phi]$ is of type 2 if $\|\phi'\|_\infty<\infty$. Using this theorem, R. Coifman-G. David-Y. Meyer showed, in [4], the following

Theorem. *If $h(t)$ is infinitely differentiable, then $C[h,\phi]$ is of type 2 as long as $\|\phi'\|_\infty<\infty$.*

The purpose of this paper is to give a new proof of this theorem. We shall deduce this theorem from Calderón's theorem and so-called "good λ inequalities". The author expresses his thanks to Prof. A. Uchiyama, through whose notebook the author learned recent Calderón's lecture on $C[\phi]$.

§2. **Proof of Theorem.** Without loss of generality we may assume that $h(t)$ has a compact support. Let $\hat{h}(\xi)=\int_{-\infty}^{\infty}e^{-i\xi t}h(t)\,dt$. Then we have formally

$$(2)\qquad C[h,\phi](x,y)=\text{Const.}\int_{-\infty}^{\infty}\hat{h}(\xi)C[e^{i\xi\cdot},\phi](x,y)\,d\xi,$$

and hence it is natural to investigate kernels $K[\psi]=C[e^{i\cdot},\psi]$ for real-valued functions $\psi(x)$. For a locally integrable function $f(x)$, we put

$K[\psi]^*f(x)=\sup\left\{\left|\int_{\varepsilon<|x-y|<\eta} K[\psi](x,y)f(y)\,dy\right|;0<\varepsilon<\eta\right\}$. We say that $K[\psi]^*$ is of weak type 1 if there exists a constant A such that, for any integrable function $f(x)$ and $\lambda>0$,

$$(3)\qquad |\{x;K[\psi]^*f(x)>\lambda\}|\leqq (A/\lambda)\|f\|_1,$$

where $|\cdot|$ denotes the 1-dimensional Lebesgue measure and $\|f\|_1=\int_{-\infty}^{\infty}|f(x)|dx$. The lower bound of such A's is denoted by $\|K[\psi]^*\|_w$. Here are two lemmas necessary for the proof; Lemma 1 is easily deduced from good λ inequalities [3].

Lemma 1. $\|K[\psi]\|_2\leqq \text{Const.}\{1+\|\psi'\|_\infty+\|K[\psi]^*\|_w\}$.

Lemma 2 (Calderón [2]). *There exists an absolute constant B such that $\|K[\psi]^*\|_w\leqq B$ as long as $\|\psi'\|_\infty\leqq 1$.*

We put $\rho(\alpha)=\sup\{\|K[\psi]^*\|_w;\|\psi'\|_\infty\leqq\alpha\}$ $(\alpha>0)$. Using good λ inequalities [3], we shall show the following inequality:

$$(4)\qquad \rho(\alpha)\leqq C\rho(p\alpha)+(C\alpha+B)\qquad(\alpha>0),$$

where $p=2/3$ and C is an absolute constant.

Once (4) is known, we have, with an absolute constant M, $\rho(\alpha)\leqq\text{Const.}(1+\alpha^M)$ $(\alpha>0)$. This inequality and Lemma 1 show that $\|K[\psi]\|_2\leqq\text{Const.}\{1+\|\psi'\|_\infty+\|\psi'\|_\infty^M\}$. The above theorem immediately follows from this inequality.

From now we prove (4). If $0<\alpha\leqq 1$, then Lemma 2 gives the required inequality. Let $\alpha>1$ and $\psi(x)$ satisfy $\|\psi'\|_\infty\leqq\alpha$. Given a real-valued integrable function $f(x)$ with compact support, we put

$$(5)\qquad U(\lambda)=\{x;K[\psi]^*f(x)>\lambda\},\quad \sigma(\lambda)=|U(\lambda)|\quad(\lambda>0).$$

We fix for a while $\lambda>0$. Since $K[\psi]^*f(x)$ is lower semi-continuous and $\lim_{|x|\to\infty}K[\psi]^*f(x)=0$, $U(\lambda)$ is an open set with finite measure. Hence we can write $U(\lambda)=\bigcup_{j=1}^\infty I_j$ with a sequence $\mathcal{M}(\lambda)=\{I_j\}_{j=1}^\infty$ of mutually disjoint finite open intervals. Let $I=(a,b)\in\mathcal{M}(\lambda)$. Then a standard argument yields the following lemma. (See for example [3].)

Lemma 3. *There exists an absolute constant C_1 such that, for any $0<\gamma\leqq 1/C_1\alpha$,*

$$(6)\qquad |x\in I;K[\psi]^*f(x)>q\lambda,\ f^*(x)\leqq\gamma\lambda|\leqq\tau_\psi(\lambda/100,\gamma\lambda)+|I|/100,$$

where $q=11/10$, $f^*(x)$ denotes the maximal function [7, p. 4] *of $f(x)$,*

$$(7)\qquad \tau_\psi(\lambda/100,\gamma\lambda)=|x\in I;K[\psi]^*(\chi f)(x)>\lambda/100,\ f^*(x)\leqq\gamma\lambda|$$

and $\chi(x)$ is the characteristic function of I.

Lemma 4. *There exists a real-valued function $\theta(x)$ with $\|\theta'\|_\infty\leqq p\alpha$ such that*

$$(8)\qquad \tau_\psi(\lambda/100,\gamma\lambda)\leqq\tau_\theta(\lambda/200,\gamma\lambda)+4|I|/5$$

as long as $0<\gamma\leqq 1/C_2\alpha$, where C_2 is an absolute constant.

Proof. Given $\gamma>0$, we may assume that $f^*(d)\leqq\gamma\lambda$ for some $d\in I$. Since $K[\psi]^*f=K[\psi-\psi(a)]^*f=K[-\psi+\psi(a)]^*f$, we may assume that $\psi(a)=0$ and $\psi(b)\geqq 0$. Put $\tilde\theta(x)=\psi(x)+\alpha(x-a)/3$. Then $\|\tilde\theta'\|_\infty\leqq 2p\alpha$,

$\tilde{\theta}(a)=0$ and $\tilde{\theta}(b)\geqq\alpha|I|/3$. Since $K[\psi]^*f=K[\tilde{\theta}]^*f$, we have $\tau_\psi(\lambda/100,\gamma\lambda)=\tau_{\tilde{\theta}}(\lambda/100,\gamma\lambda)$. We define $\theta^*(x)$ by "*the running water*" of $\theta(x)$:

$$\text{(9)}\qquad \theta^*(x)=\begin{cases}0 & (x<a)\\ \inf\{\phi(x)\,;\,\phi\geqq\tilde{\theta} \text{ and } \phi'\geqq 0 \text{ on } [a,b]\} & (a\leqq x\leqq b)\\ \theta^*(b) & (x>b).\end{cases}$$

Then $\theta^*(x)$ is a non-decreasing continuous function satisfying $\{x\in I\,;\,\theta^*(x)>\tilde{\theta}(x)\}\subset\{x\in I\,;\,\theta^{*\prime}(x)=0\}$. Since $\|\theta^{*\prime}\|_\infty\leqq 2p\alpha$, $\theta^*(a)=0$ and $\theta^*(b)\geqq\alpha|I|/3$, we have $|V|\geqq|I|/4$, where $V=\{x\in I\,;\,\theta^*(x)=\tilde{\theta}(x)\}$. For any $y\in I-V$, we have $|\tilde{\theta}(y)-\theta^*(y)|\leqq 2\|\tilde{\theta}'\|_\infty \operatorname{dis}(y,V)\leqq 4p\alpha \operatorname{dis}(y,V)$, where $\operatorname{dis}(y,V)$ denotes the distance between y and V. Hence, for any $x\in V$,

$$\text{(10)}\quad \int_{-\infty}^{\infty}|K[\tilde{\theta}](x,y)-K[\theta^*](x,y)||(\chi f)(y)|dy \leqq 4p\alpha\int_{-\infty}^{\infty}\{\operatorname{dis}(y,V)/(x-y)^2\}|(\chi f)(y)|dy \quad (=4p\alpha M(x),\text{ say}).$$

Now we put $\theta(x)=\theta^*(x)-p\alpha x$. Then $\|\theta'\|_\infty\leqq p\alpha$ and $K[\theta]^*f=K[\theta^*]^*f$. Thus

$$\begin{aligned}\text{(11)}\qquad \tau_\psi(\lambda/100,\gamma\lambda)&=\tau_{\tilde{\theta}}(\lambda/100,\gamma\lambda)\\ &\leqq|x\in V\,;\,K[\tilde{\theta}]^*(\chi f)(x)>\lambda/100,\ f^*(x)\leqq\gamma\lambda|+|I-V|\\ &\leqq|x\in V\,;\,K[\theta^*]^*(\chi f)(x)>\lambda/200,\ f^*(x)\leqq\gamma\lambda|\\ &\quad+|x\in V\,;\,4p\alpha M(x)>\lambda/200|+3|I|/4\\ &\leqq\tau_\theta(\lambda/200,\gamma\lambda)+|x\in V\,;\,4p\alpha M(x)>\lambda/200|+3|I|/4.\end{aligned}$$

Let us recall $f^*(d)\leqq\gamma\lambda$. Since

$$4p\alpha\int_V M(x)dx\leqq 4p\alpha\|\chi f\|_1\leqq\text{Const. }\alpha f^*(d)|I|\leqq\text{Const. }\alpha\gamma\lambda|I|,$$

there exists an absolute constant C_2 such that $|x\in V\,;\,4p\alpha M(x)>\lambda/200|\leqq(C_2/100)\alpha\gamma|I|$. Hence (11) gives (8) as long as $0<\gamma\leqq 1/C_2\alpha$. Q.E.D.

By Lemmas 3 and 4, we have

$$|x\in I\,;\,K[\psi]^*f(x)>q\lambda,\ f^*(x)\leqq\gamma\lambda| \leqq\tau_\psi(\lambda/100,\gamma\lambda)+|I|/100\leqq\tau_\theta(\lambda/200,\gamma\lambda)+5|I|/6$$

as long as $0<\gamma\leqq 1/C_3\alpha$, where $C_3=\max\{C_1,C_2\}$. If $f^*(x)>\gamma\lambda$ for all $x\in I$, then $\tau_\theta(\lambda/200,\gamma\lambda)=0$. If $f^*(d)\leqq\gamma\lambda$ for some $d\in I$, then we have, with an absolute constant C_4,

$$\tau_\theta(\lambda/200,\gamma\lambda)\leqq\{200\rho(\|\theta'\|_\infty)/\lambda\}\|\chi f\|_1\leqq\{C_4\rho(p\alpha)/\lambda\}f^*(d)|I|\leqq C_4\gamma\rho(p\alpha)|I|.$$

Let $\gamma_0=1/\{C_3\alpha+100C_4\rho(p\alpha)\}$. Then we have, for any $I\in\mathcal{M}(\lambda)$,

$$|x\in I\,;\,K[\psi]^*f(x)>q\lambda,\ f^*(x)\leqq\gamma_0\lambda|\leqq r|I|,$$

where $r=6/7$. Taking the summation over all I in $\mathcal{M}(\lambda)$, we have

$$|x\,;\,K[\psi]^*f(x)>q\lambda,\ f^*(x)\leqq\gamma_0\lambda|\leqq r\sigma(\lambda).$$

Hence

$$\text{(12)}\qquad \sigma(q\lambda)\leqq\kappa(\gamma_0\lambda)+r\sigma(\lambda),$$

where $\kappa(\gamma_0\lambda)=|x\,;\,f^*(x)>\gamma_0\lambda|$. Note that $\kappa(\gamma_0\lambda)\leqq\{\text{Const.}/\gamma_0\lambda\}\|f\|_1$ ([7, p. 5]). Inequality (12) is valid with λ replaced by λ/q^k $(k\geqq 1)$. Hence

$$\sigma(\lambda)\leqq\lim_{n\to\infty}\left\{\sum_{k=1}^{n} r^k\kappa(\gamma_0\lambda/q^k)+r^{n+1}\sigma(\lambda/q^{n+1})\right\}\leqq\left\{\text{Const.}\sum_{k=1}^{\infty}(rq)^k\right\}\|f\|_1/\gamma_0\lambda.$$

Since $\|K[\psi]^*\|_w$ is dominated by the upper bound of $2\lambda|x\,;K[\psi]^*f(x)>\lambda|/\|f\|_1$ over all $\lambda>0$ and all real-valued integrable functions $f(x)$ with compact support, we have, with an absolute constant C, $\|K[\psi]^*\|_w\leqq\text{Const.}/\gamma_0\leqq C\rho(p\alpha)+(C\alpha+B)$. Since $\psi(x)$ is arbitrary as long as $\|\psi'\|_\infty\leqq\alpha$, we have (4). This completes the proof.

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