

**104. Non-Uniqueness in the Cauchy Problem for Partial Differential Operators with Multiple Characteristics. II**

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(Communicated by Kôzaku YOSIDA, M. J. A., Oct. 12, 1983)

In this note, we shall consider non-characteristic Cauchy problem for a class of partial differential operators with  $C^\infty$ -coefficients containing some degenerate elliptic operators. Our purpose is to give necessary conditions for uniqueness on the lower order terms of operators. And we shall extend the results of our preceding papers [1]–[3].

Let  $U$  be an open neighborhood of the origin in  $\mathbb{R}^{d+1}$ . And let  $P = P(t, x; \partial_t, D_x)$  be the following operator of order  $p$  in  $U$ :

$$(1) \quad P = (\partial_t - t^k C(t, x; D_x))^p + t^l A(t, x; D_x) - t^m B(t, x; D_x) + \sum_{j=1}^p \sum_{i \leq j} t^{m(j,i)} B_{j,i}(t, x; D_x) \partial_t^{p-j},$$

where

$\partial_t = \partial/\partial t$ ,  $D_x = -\sqrt{-1}(\partial/\partial x_1, \dots, \partial/\partial x_d)$ ,  $k, l, m, m(j, i) \in \mathbb{N} = \{0, 1, 2, \dots\}$ ,  $A, B, C, B_{j,i}$  are partial differential operators, homogeneous order  $q, q-r, 1, i$  with respect to  $D_x$  respectively, whose coefficients are  $C^\infty$  in  $U$  and  $p \geq q > r \geq 1$ .

Then the following theorem is a corollary of Theorem 1.1 of Nakane [2].

**Theorem 1.** *Suppose*

$$(2) \quad \frac{pr + qm}{q - r} < k < \frac{prl + (p - q)m}{p - q + r},$$

$$(3) \quad m(j, i) > \frac{jk}{p} + \frac{(ip - jq)(k - m)}{pr}.$$

We also assume that there exist  $\xi^0 \in \mathbb{R}^d \setminus \{0\}$  and a branch  $D(\xi^0)$  of  $(B(0, 0; \xi^0) - A(0, 0; \xi^0))^{1/p}$  satisfying

$$(4) \quad \operatorname{Re} D(\xi^0) > 0,$$

$$(5) \quad \operatorname{Re} \left\{ \left( \frac{A(0, 0; \xi^0)}{B(0, 0; \xi^0) - A(0, 0; \xi^0)} + 1 - \frac{q}{r} \right) D(\xi^0) \right\} > 0.$$

Then there exist an open neighborhood  $U'$  of the origin in  $\mathbb{R}^{d+1}$  and  $C^\infty$ -functions  $u, f$  in  $U'$  such that

$$(6) \quad Pu - fu = 0, \quad (0, 0) \in \operatorname{supp} u \subset \{t \geq 0\}.$$

**Remark 1.** Assumption (2) is equivalent to assumption (1.9) or (1.13) of Theorem 2 of Nakane [1]. Hence this theorem is a generalization of Theorem 2 of [1].

**Remark 2.** As in Remark 1.4 of [2], we introduce Newton polygons. In the  $(X, Y)$ -plane, we plot the following points:

$$R_1=(q/p, -1), \quad R_2=(0, k/p), \quad R_3=(r/p, m/p),$$

$$R_4=(q/p-1, l), \quad P_{j,i}=(q/p-i/j, m(j, i)/j).$$

The first inequality of (2) implies that  $R_3$  is located below the line passing through the points  $R_1$  and  $R_2$ . The second inequality of (2) implies that  $R_4$  is located above the line passing through the points  $R_2$  and  $R_3$ . Assumption (3) implies that all the points  $P_{j,i}$  are located above the line passing through the points  $R_2$  and  $R_3$ . Hence above theorem is a corollary of Theorem 1. 1 of [2].

Now we consider the case  $k > \frac{prl+(p-q)m}{p-q+r}$ .

**Theorem 2.** Suppose

$$(7) \quad k > \frac{prl+(p-q)m}{p-q+r},$$

$$(8) \quad m < (l+1)(q-r)-p,$$

$$(9) \quad m(j, i) > lj + \frac{m-pl}{p-q+r}(j-i).$$

We also assume that there exist  $\xi^0 \in \mathbf{R}^d \setminus \{0\}$  and a branch  $B(0, 0; \xi^0)^{1/p}$  satisfying

$$(10) \quad \operatorname{Re} C(0, 0; \xi^0) + \operatorname{Re} B(0, 0; \xi^0)^{1/p} > 0,$$

$$(11) \quad p \operatorname{Re} C(0, 0; \xi^0) + (q-r) \operatorname{Re} B(0, 0; \xi^0)^{1/p} < 0.$$

Then the same conclusion as in Theorem 1 holds.

**Remark 3.** In terms of Remark 2, assumption (7) implies that  $R_4$  is located below the line passing through the points  $R_2$  and  $R_3$ . Assumption (8) implies that  $R_3$  is located below the line passing through the points  $R_1$  and  $R_4$ . Assumption (9) implies that all the points  $P_{j,i}$  are located above the line passing through the points  $R_3$  and  $R_4$ .

**Example.** We consider the following operator in  $\mathbf{R}^2$ :

$$P = (\partial_t - t^l D_x)^p - t^m B(t, x) D_x^{q-r},$$

where  $B \in C^\infty(U)$  and  $p \geq q > r \geq 1$ . Since it corresponds to the case  $A = B_{j,i} = 0$ , assumptions (7) and (9) are automatically satisfied. We assume (8) and we consider assumptions (10) and (11). By considering the effect of the similarity transformation:  $x \rightarrow hx$  for some  $h \in \mathbf{R}$ , we have the following.

*Case 1.* When  $p \geq 3$ , assumptions (10) and (11) are satisfied if  $B(0, 0) \neq 0$ .

*Case 2.* When  $p = q = 2$  and  $r = 1$ , assumptions (10) and (11) are satisfied if  $B(0, 0) \in C \setminus [0, \infty)$ .

**Remark 4.** Consider the following operator in  $\mathbf{R}^2$ :

$$P = (\partial_t - t^l D_x)^2 - t^m B(t, x) D_x + C(t, x),$$

where  $B, C \in C^\infty(U)$ . In [1], we showed that uniqueness holds for  $P$  if  $m > l - 1$ . Recently Ōkaji [4] showed that uniqueness holds for  $P$  if  $m \geq l - 1$ . Furthermore, Prof. K. Watanabe pointed us that uniqueness holds for  $P$  with  $m < l - 1$  if  $B(t, x) > 0$ . Hence assumptions (8), (10) and (11) are indispensable.

Finally we consider the case  $k = \frac{prl + (p - q)m}{p - q + r}$ .

**Theorem 3.** *Suppose (8), (9) and*

$$(12) \quad k = \frac{prl + (p - q)m}{p - q + r}.$$

*We also assume that there exist  $\xi^0 \in \mathbf{R}^d \setminus \{0\}$  and a branch  $D(\xi^0)$  of  $(B(0, 0; \xi^0) - A(0, 0; \xi^0))^{1/p}$  satisfying*

$$(13) \quad \operatorname{Re} C(0, 0; \xi^0) + \operatorname{Re} D(\xi^0) > 0,$$

$$(14) \quad p \operatorname{Re} C(0, 0; \xi^0) + \operatorname{Re} \left\{ \left( \frac{(p - q + r)(m - k)A(0, 0; \xi^0)}{B(0, 0; \xi^0) - A(0, 0; \xi^0)} + q - r \right) D(\xi^0) \right\} < 0.$$

*Then the same conclusion as in Theorem 1 holds.*

**Remark 5.** Assumption (12) implies that  $R_4$  is located on the line passing through the points  $R_2$  and  $R_3$ .

### References

- [1] S. Nakane: Uniqueness and non-uniqueness in the Cauchy problem for a class of operators of degenerate type (to appear in *J. Differential Equations*).
- [2] —: Non-uniqueness in the Cauchy problem for partial differential operators with multiple characteristics I (submitted to *Comm. in P. D. E.*).
- [3] —: Non-uniqueness in the Cauchy problem for partial differential operators with multiple characteristics. *Proc. Japan Acad.*, 59A, 41–43 (1983).
- [4] T. Ōkaji: Uniqueness in the Cauchy problem for a class of partial differential operators degenerate on the initial surface (preprint).