97. On v-Ideals in a VHC Order*)

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Throughout this note, Q will be a simple artinian ring and R will be an order in Q with 1. Let $\underline{C}$ ($\underline{C}'$) be a right (left) Gabriel topology on R cogenerated by the right (left) injective hull of Q/R. In [4], R is called a *VH* (*v-hereditary*) *order* if for any R-ideal A such that $_vA = A$ ($A_v = A$) we have $_v(A(R:A)_l) = O_l(A)$ (resp. $((R:A)_rA)_v = O_r(A)$). We say that R is a *VHC order* if it is a VH order satisfying the maximum condition on $\underline{C}$-closed right ideals and $\underline{C}'$-closed left ideals. The concept of VHC orders is a Krull type generalization of HNP (hereditary noetherian prime) rings. The aim of this note is to extend Robson's theorems and Fujita-Nishida's theorems in HNP rings to the case of VHC orders (cf. [1], [7] and [3]). Concerning our terminology and notations we refer to [4]. See [6] for many interesting examples of VHC orders.

Proposition 1. *The following two conditions are equivalent*:

(1) $_v(A(R:A)_l) = O_l(A)$ *for any R-ideal A such that* $_vA = A$.

(2) $_v(A(R:A)_l) = {}_v(O_l(A))$ *for any R-ideal A.*

Proof. (2)$\Rightarrow$(1) is clear, because $_v(O_l(A)) = O_l(A)$ for any R-ideal A with $_vA = A$. (1)$\Rightarrow$(2): Since $_vA \supset A$, we have $1 \in O_l(_vA) = {}_v(_vA(R:{}_vA)_l) \subset {}_v(_vA(R:A)_l) = {}_v(A(R:A)_l)$ by Lemma 1.1 of [4]. It is clear that $A(R:A)_l \subset O_l(A)$ and so $_v(A(R:A)_l) \subset {}_v(O_l(A))$. On the other hand, $A(R:A)_l$ is an $(O_l(A), O_l(A))$-bimodule and thus $_v(A(R:A)_l)$ is a right $O_l(A)$-module. Hence it follows that $O_l(A) \subset {}_v(A(R:A)_l)$ and that $_v(O_l(A)) \subset {}_v(A(R:A)_l)$.

From now on, *R will be a VHC order in a simple artinian ring Q.*

Lemma 1. *Let A be any R-ideal. Then* $_vA = A_v$.

Proof. This is proved as in Lemma 1.2 of [4] by using Proposition 1.

We consider the following sets of v-ideals of R: $V(R) = \{A : \text{ideal of } R \mid A : v\text{-ideal}\} \supset V_m(R) = \{A \in V(R) \mid A \subset P : \text{prime } v\text{-ideal} \Rightarrow P : \text{maximal } v\text{-ideal}\}$. If R has enough v-invertible ideals, then $V(R) = V_m(R)$ by Lemma 1.2 of [5]. We do not have an example of VHC order in which $V(R) \supsetneqq V_m(R)$ up to now. We study the properties of ideals belonging to $V_m(R)$.

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Proposition 2. (1) *If $A, B \in V_m(R)$, then $AB \in V_m(R)$.*

(2) *Let A and B be elements in $V(R)$ such that $A \subset B$. If $A \in V_m(R)$, then $B \in V_m(R)$.*

(3) *If $A \in V_m(R)$, then $Ass(R/A)$ consists of maximal v-ideals of R.*

(4) *Let X be any v-invertible ideal of R. Then $X \in V_m(R)$.*

(5) *Let A be any element in $V(R)$. Then $A \in V_m(R)$ if and only if there are maximal v-ideals $M_1, \cdots, M_n$ satisfying $M_1 \cdots M_n \subset A \subset M_i$ for any $i=1, \cdots, n$.*

Proof. (1), (2) and (3) are trivial. (4): As in Propositions 2.10 and 2.11 of [4], we have $R = \cap R_P \cap S(R)$, where R_P is an HNP ring whose Jacobson radical $P' = PR_P = R_P P$ is a unique maximal invertible ideal of R_P (P ranges over all maximal v-invertible ideals of R), $S = S(R) = \cup Y^{-1}$ (Y runs over all v-invertible ideals of R), and $(XS)_v = S = (SX)_v$. Now let A be a prime v-ideal containing X. Then we have $A = \cap AR_P \cap (AS)_v = \cap AR_P \cap S$. There are only a finite number of maximal v-invertible ideals $P_1, \cdots, P_n$ of R such that $R_{P_i} \supsetneqq AR_{P_i}$ $(1 \leqq i \leqq n)$ and so $A = A_1 \cap \cdots \cap A_n$ $(A_i = AR_{P_i} \cap R)$. Since A is a prime ideal, we have $A = A_i$ for some i and so AR_{P_i} is also a prime ideal. Write $P_i = M_1 \cap \cdots \cap M_k$, an intersection of a cycle, where M_j are maximal v-ideals of R. Then $\{M_j R_{P_i} \mid 1 \leqq j \leqq k\}$ are only prime ideals of R_{P_i} (see Proposition 2.7 of [4]). Thus $AR_{P_i} = M_j R_{P_i}$ for some j and $A = AR_{P_i} \cap R = M_j$, a maximal v-ideal of R. Since R satisfies a.c.c. on v-ideals of R, (5) easily follows (see the proof of Lemma 1.2 of [8]).

Proposition 3. (1) *Let A be any element in $V_m(R)$. Then $A = (XB)_v$ for some v-invertible ideal X of R and eventually v-idempotent ideal $B \in V_m(R)$.*

(2) *Let C be an eventually v-idempotent ideal in $V_m(R)$ and let $M_1, \cdots, M_n$ be the full set of maximal v-ideals containing C. Then $(C^n)_v = ((M_1 \cap \cdots \cap M_n)^n)_v$ and is v-idempotent.*

Proof. (1) As in Theorem 4.2 of [1]. (2) follows from the proof of Proposition 1.4 of [6].

Lemma 2. *Let M_1 and M_2 be any maximal v-ideals of R such that $O_r(M_i) \neq O_l(M_j)$ for all i, j $(1 \leqq i, j \leqq 2)$ and let $A = M_1 \cap M_2$. Then $A = (M_1 M_2)_v = (M_2 M_1)_v$ and is v-idempotent.*

Proof. First we note that $A \in V_m(R)$. Assume that A is not v-idempotent. Then, by Lemma 1.3 of [6], we have $R \supsetneqq (A(R:A)_r)_v \supsetneqq A$ and $R \supsetneqq ((R:A)_l A)_v \supsetneqq A$, because $((R:A)_l A)_v$ and $(A(R:A)_r)_v$ are both v-idempotent. So we may assume that $((R:A)_l A)_v = M_1$ by Propositions 2 and 3, and then $A = (M_2 M_1)_v$ by Lemma 1.3 of [6]. Thus we have $O_r(A) = O_r(M_1)$. Assume that $M_1 = (A(R:A)_r)_v$. Then $O_l(M_1) \supset O_l(A) \supset O_l(M_2)$ and so $M_1 \subset M_2$. This is a contradiction. Hence M_2

$=(A(R:A)_r)_v$. Now assume that $W=O_r(M_1)\cap O_l(M_2)\supsetneqq R$. Then $R\supsetneqq(R:W)_r\supset(R:O_l(M_2))_r=M_2$ and so $(R:W)_r=M_2$. Similarly, we have $(R:W)_l=M_1$. Thus $O_r(M_1)=W_v={}_vW=O_l(M_2)$ by Lemma 1. This is a contradiction. Hence $O_r(M_1)\cap O_l(M_2)=R$. On the other hand, since $(A^2)_v$ is v-idempotent by Lemma 1.3 of [6], we have $K=O_r((A^2)_v)\cap O_l((A^2)_v)\supsetneqq R$ by the same method as in Lemma 1.7 of [6]. The inclusions $(A^2)_v\subset(R:K)_l\subsetneqq R$ imply that $(R:K)_l$ is contained in a maximal v-ideal of R, say M_1. Then $K_v={}_vK\supset O_r(M_1)\supsetneqq R$. This entails that $O_r(M_1)$ is a v-ideal. So it follows from Lemma 1.7 of [2] that there exists a v-idempotent ideal N containing $(A^2)_v$ such that $O_r(M_1)=O_l(N)$. Since $O_r(M_1)$ is minimal in the set of all overrings of R which are v-ideals, N must be a maximal v-ideal of R and thus $N=M_2$, which is a contradiction. Therefore A must be v-idempotent.

Distinct v-idempotent, maximal v-ideals $M_1, \cdots, M_n$ are called an *open cycle* if $O_r(M_1)=O_l(M_2), \cdots, O_r(M_{n-1})=O_l(M_n)$ but $O_r(M_n)\neq O_l(M_1)$. The following proposition is due to Fujita and Nishida if R is an HNP ring which is obtained in a similar way to prove Theorem 1.3 of [3] by using Lemma 1.3 of [6], Propositions 2, 3 and Lemma 2.

Proposition 4. *Let $M_1, \cdots, M_n$ be an open cycle and let $A=M_1\cap\cdots\cap M_n$. Then*

(1) $(A(R:A)_r)_v=M_1$ *and* $((R:A)_lA)_v=M_n$.

(2) $A=(M_1\cdots M_n)_v$.

(3) $(AM_i)_v=(M_{i+1}A)_v$ *for* $i=1, \cdots, n-1$.

(4) $(A^i((R:A)_r)^i)_v=(M_i\cdots M_1)_v$ *and* $(((R:A)_l)^iA^i)_v=(M_n\cdots M_{n-i+1})_v$. *In particular,* $(A^n)_v=(A^n((R:A)_r)^n)_v=(((R:A)_l)^nA^n)_v=(M_n\cdots M_1)_v$.

(5) $A\supsetneqq(A^2)_v\supsetneqq\cdots\supsetneqq(A^n)_v=(A^{n+1})_v=\cdots$.

Let $M_1, \cdots, M_m$ and $N_1, \cdots, N_n$ be distinct v-idempotent, maximal v-ideals of R. Then, following [3], $M_1, \cdots, M_m$ and $N_1, \cdots N_n$ are *separated* if $O_r(M_i)\neq O_l(N_j)$ and $O_r(N_j)\neq O_l(M_i)$ for all $i=1, \cdots, m$ and $j=1, \cdots, n$. Proposition 3 allows us to study v-invertible ideals and eventually v-idempotent ideals separately. The structure of v-invertible ideals was completely determined in [4] (see Theorem 1.13 of [4]). To study eventually v-idempotent ideals of R, let $M_1, \cdots, M_n$ be a finite set of distinct v-idempotent, maximal v-ideals of R such that $A=M_1\cap\cdots\cap M_n$ is not contained in any v-invertible ideals of R (see Proposition 3). Then we classify it as follows;

(a) $\{M_1, \cdots, M_n\}=\bigcup_{i=1}^k\{M_{i1}, \cdots, M_{in(i)}\}$, and each of $M_{i1}, \cdots, M_{in(i)}$ is an open cycle.

(b) $M_{i1}, \cdots, M_{in(i)}$ and $M_{j1}, \cdots, M_{jn(j)}$ are separated for any i, j $(i\neq j)$. Put $A_i=M_{i1}\cap\cdots\cap M_{in(i)}$. Then we have

Proposition 5. *With the above notations and assumption we* *have* $A=(A_1\cdots A_k)_v$ *and* $(A_iA_j)_v=(A_jA_i)_v$ (*cf.* [3]).

Proof. By Proposition 4, $A_i=(M_{i1}\cdots M_{in(i)})_v$ and so $(A_iA_j)_v=(A_jA_i)_v$ by Lemma 2. We shall prove $A=(A_1\cdots A_k)_v$ by induction on k. If $k=1$, then there is nothing to prove. So we may assume that $B=A_1\cap\cdots\cap A_{k-1}=(A_1\cdots A_{k-1})_v$. Then $(BA_k)_v=(A_kB)_v$ by Lemma 2 and $(B+A_k)_v=R$. Thus $A=B\cap A_k=((B\cap A_k)(B+A_k)_v)_v\subset(BA_k)_v+(A_kB)_v=(BA_k)_v=(A_1\cdots A_k)_v$ and therefore $A=(A_1\cdots A_k)_v$.

The next proposition is due to Robson in case R is an HNP ring (see [7]) and the author obtained the proposition if R is a VHC order with enough v-invertible ideals (see [6]).

Proposition 6. *Let* $M_1,\cdots,M_n$ *be maximal v-ideals of* R *and let* $A=M_1\cap\cdots\cap M_n$. *Then* A *is v-idempotent if and only if* $O_r(M_i)\neq O_l(M_j)$ *for any* i, j.

Proof. Assume that A is v-idempotent and that $O_r(M_i)=O_l(M_j)$ for some i, j. If $i=j$, then M_i is v-invertible and so $A\subset\bigcap_{n=1}^{\infty}(M_i^n)_v=O$, a contradiction. Hence $i\neq j$. Let $A=(A_1\cdots A_k)_v$ be the decomposition of A as in Proposition 5. Then there exists A_i, say A_1, such that $A_1=M_{11}\cap\cdots\cap M_{1n(1)}$ with $n(1)\geqq 2$. Then we have, by Proposition 4, ${}_v(M_{1n(1)}A_2\cdots A_k)={}_v((R:A)_lA_1A_2\cdots A_k)={}_v((R:A_1)_lA_1^2A_2^2\cdots A_k^2)={}_v(M_{1n(1)}A_1A_2^2\cdots A_k^2)\subset M_{11}$, which is a contradiction. Hence $O_r(M_i)\neq O_l(M_j)$ for all i, j. We prove the sufficiency by induction on n (see Lemma 2 in case $n=2$). So we may assume that $B=M_1\cap\cdots\cap M_{n-1}=(M_1\cdots M_{n-1})_v$ is v-idempotent and $(B+M_n)_v=R$. Thus $A=B\cap M_n=((B\cap M_n)(B+M_n)_v)_v\subset((BM_n)_v+(M_nB)_v)_v=(M_1\cdots M_n)_v$ by Lemma 2. Hence $A=(M_1\cdots M_n)_v$ and is v-idempotent, because $(M_iM_j)_v=(M_jM_i)_v$.

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