11. *Integrality of Certain Algebraic Values Attached to Modular Forms*

By Shin-ichiro MIZUMOTO
Department of Mathematics, Tokyo Institute of Technology

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Introduction. In this note we report integrality results on some algebraic numbers appearing in the theory of modular forms: Quotients of Petersson inner products for elliptic modular forms, special values of the "second" L-function attached to elliptic eigen cusp forms, and Fourier coefficients of generalized Eisenstein series of degree two in the sense of Langlands and Klingen. Details will appear elsewhere. For motivations, we refer to Kurokawa [4], [5] and [6], [8], [9]. The author would like to thank Prof. N. Kurokawa for encouragements.

§ 1. **Petersson inner products.** For integers $n \geqq 1$ and $k \geqq 0$, we denote by $M_k(\Gamma_n)$ (or $S_k(\Gamma_n)$) the vector space over the complex number field $\boldsymbol{C}$ consisting of all Siegel modular (or cusp) forms of degree n and weight k. Each $F \in M_k(\Gamma_n)$ has a Fourier expansion of the form: $F=\sum_{T \geqq 0} a(T, F)q^T$, where $q^T=\exp(2\pi\sqrt{-1}\operatorname{trace}(TZ))$ with a variable Z on the Siegel upper half space of degree n, and T runs over all $n \times n$ symmetric positive semi-definite semi-integral matrices. For a subring R of $\boldsymbol{C}$, we put $M_k(\Gamma_n)_R=\{F \in M_k(\Gamma_n) \mid a(T, F) \in R \text{ for all } T \geqq 0\}$ and $S_k(\Gamma_n)_R=S_k(\Gamma_n) \cap M_k(\Gamma_n)_R$ (R-modules).

In the first two sections, we treat elliptic modular forms. For each integer $m \geqq 1$, let $T(m): M_k(\Gamma_1) \to M_k(\Gamma_1)$ be the m-th Hecke operator and $\boldsymbol{T}_{\boldsymbol{Q}}=\boldsymbol{Q}[T(m) \mid m \geqq 1]$ be the Hecke algebra over the rational number field $\boldsymbol{Q}$. For $F \in S_k(\Gamma_1)$ and $G \in M_k(\Gamma_1)$ we put

$$\langle F, G\rangle=\frac{3}{\pi}\int_{\mathfrak{F}} F(z)\overline{G(z)}y^{k-2}dxdy \qquad (z=x+\sqrt{-1}\,y)$$

where $\mathfrak{F}$ is a fundamental domain of $\Gamma_1 \backslash \mathfrak{H}_1$, $\mathfrak{H}_1$ being the upper half plane.

Let $\bar{\boldsymbol{Q}}$ be the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$, and let $\bar{\boldsymbol{Z}}$ be the ring of algebraic integers in $\bar{\boldsymbol{Q}}$. For each non-zero $\alpha \in \bar{\boldsymbol{Q}}$, we denote by $\mathrm{D}(\alpha)$ the minimal positive $M \in \boldsymbol{Z}$ $(=\bar{\boldsymbol{Z}} \cap \boldsymbol{Q})$ such that $M\alpha \in \bar{\boldsymbol{Z}}$, and we put $\operatorname{Num}(\alpha)=\mathrm{D}(\alpha)\alpha$.

Now let $f=\sum_{n \geqq 1} a(n)q^n \in S_k(\Gamma_1)$ be a normalized eigen cusp form, i.e., $a(1)=1$ and $T(n)f=a(n)f$ for all $n \geqq 1$. Let $\boldsymbol{Q}(f)=\boldsymbol{Q}(a(n) \mid n \geqq 1)$ be the totally real number field generated by the eigen values of f, and let $\mathfrak{D}(\boldsymbol{Q}(f))$ be the different of $\boldsymbol{Q}(f)/\boldsymbol{Q}$. We put $\boldsymbol{Z}(f)=\bar{\boldsymbol{Z}} \cap \boldsymbol{Q}(f)$, and denote by κ the exponent of the finite abelian group $\boldsymbol{Z}(f)/\boldsymbol{Z}[a(n) \mid n \geqq 1]$.

Suppose $V\subset S_k(\Gamma_1)_{\boldsymbol{Q}}$ is a $\boldsymbol{T_Q}$-irreducible $\boldsymbol{Q}$-subspace such that $V\otimes_{\boldsymbol{Q}}\boldsymbol{C}\ni f$, and $V^{\perp}$ is its orthogonal complement with respect to $\langle\ ,\ \rangle$ in $S_k(\Gamma_1)_{\boldsymbol{Q}}$. Writing $V_{\boldsymbol{Z}}=V\cap S_k(\Gamma_1)_{\boldsymbol{Z}}$ and $V_{\boldsymbol{Z}}^{\perp}=V^{\perp}\cap S_k(\Gamma_1)_{\boldsymbol{Z}}$, we denote by ν the exponent of the finite abelian group $S_k(\Gamma_1)_{\boldsymbol{Z}}/(V_{\boldsymbol{Z}}\oplus V_{\boldsymbol{Z}}^{\perp})$. Note that: If

(*) $\boldsymbol{T_Q}$ acts irreducibly on $S_k(\Gamma_1)_{\boldsymbol{Q}}$,

then $\nu=1$. The condition (*) is verified at least for $k\leqq 200$ by Buhler as noted in Kohnen [2].

Theorem 1. *Let the notation be as above. For each* $g=\sum_{n\geqq 0}b(n)q^n\in M_k(\Gamma_1)_{\boldsymbol{Z}}$ *we have the following.*

(1) *Put* $\gamma=1$ *or* $\mathrm{Num}\,(B_k/k)$ *according as* $b(0)=0$ *or* $b(0)\neq 0$; *here,* B_k *is the k-th Bernoulli number. Let t be the* g.c.d. *of* $\mathrm{D}\,(B_k/2k)$ *and* $b(1), b(2),\cdots$. *Then*: $\langle f, g\rangle/\langle f, f\rangle\in(t/\gamma\kappa\nu\mathfrak{D}(\boldsymbol{Q}(f)))\boldsymbol{Z}(f)$.

(2) *Assume* (*) *holds. Let* $m\in\boldsymbol{Z}$ *be a positive integer dividing* $\mathrm{Num}\,(B_k/k)$. *Suppose* g.c.d. $(m, b(0))=1$. *Then*: $\mathrm{D}\,(\langle f, g\rangle/\langle f, f\rangle)\equiv 0 \pmod m$.

Example 1. ($k=24$) For even $w\geqq 4$, we put $E_w=1-(2w/B_w)\sum_{n\geqq 1}(\sum_{d|n,d>0}d^{w-1})q^n\in M_w(\Gamma_1)$. Let $\Delta_{12}\in S_{12}(\Gamma_1)$ be the normalized eigen cusp form of weight 12. Then, $f=\Delta_{24}^{(+)}=E_{12}\Delta_{12}+12(27017/691+\sqrt{144169})\Delta_{12}^2$ is a normalized eigen cusp form of weight 24. We have: $\boldsymbol{Q}(f)=\boldsymbol{Q}(\sqrt{144169})$, $\mathfrak{D}(\boldsymbol{Q}(f))=(\sqrt{144169})$, $\kappa=24$, and $\nu=1$.

(i) $\langle f, \Delta_{12}^2\rangle/\langle f, f\rangle=1/24\sqrt{144169}$. Thus Theorem 1(1) gives the "minimal" denominator in this case.

(ii) For $(E_4)^6=(E_6)^3$, we have $t=2^5 3^2 5$ and $\gamma=\mathrm{Num}\,(B_{24}/24)=103\cdot 2294797$. Theorem 1(1) asserts:

$$\langle f, (E_4)^6\rangle/\langle f, f\rangle\in\frac{2^2 3\cdot 5}{103\cdot 2294797\sqrt{144169}}\boldsymbol{Z}(f).$$

Moreover by Theorem 1(2), $\mathrm{Num}\,(B_{24}/24)$ appears in the denominator of $\langle f, (E_4)^6\rangle/\langle f, f\rangle$. In fact, $\langle f, (E_4)^6\rangle/\langle f, f\rangle=2^9 3^3 5^4(107957+19697\sqrt{144169})/(103\cdot 2294797\sqrt{144169})$.

§2. Special values of the second *L*-functions. For a normalized eigen cusp form $f=\sum_{n\geqq 1}a(n)q^n\in S_k(\Gamma_1)$, we put $L_2(s, f)=\prod_p(1-\alpha_p^2p^{-s})^{-1}\cdot(1-\alpha_p\beta_pp^{-s})^{-1}(1-\beta_p^2p^{-s})^{-1}$ where p runs over all prime numbers and $\alpha_p, \beta_p\in\boldsymbol{C}$ are taken so that $\alpha_p+\beta_p=a(p)$ and $\alpha_p\beta_p=p^{k-1}$. Further we put $L_2^*(s, f)=L_2(s, f)(2\pi)^{-(2s-k+2)}\Gamma(s)\langle f, f\rangle^{-1}$.

Theorem 2. *Let the notation be as in Theorem* 1. *Let* $f\in S_k(\Gamma_1)$ *be a normalized eigen cusp form.*

(1) *Let r be an odd integer such that* $1\leqq r\leqq k-1$. *Put*

$$\mathfrak{a}_r=\mathrm{Num}\left(\frac{3\cdot(k-2)!}{2^{k-6}\cdot(k-r-1)!}\right)\xi_r\kappa\nu\gamma_r\mathfrak{D}(\boldsymbol{Q}(f))$$

$$\textit{where } \xi_r=\prod_{(p-1)|2r,\ p:\ \textit{odd primes}}p[r]_p$$

with $[r]_p$ *denoting the p-primary part of r*; $\gamma_r=1$ *for* $1\leqq r\leqq k-3$ *and* $\gamma_{k-1}=\mathrm{Num}\,(B_k/k)$. *Then we have*: $L_2^*(r+k-1, f)\in(2/\mathfrak{a}_r)\boldsymbol{Z}(f)$.

(2) *Assume* $(*)$ *in* §1 *holds. Let* b_k *be the maximal divisor of* $\mathrm{Num}(B_k/k)$ *satisfying* $(b_k, \mathrm{Num}(B_{2k-2}))=1$. *Put* $\mathcal{P}(k)=\{p \in \boldsymbol{Z} \mid p \geqq 5$ *prime*, $(p-1)\mid(2k-2)$, $p \nmid \mathrm{Num}(1+(B_k/k))b_k\}$. *Then*: $\mathrm{D}(L_2^*(2k-2, f)) \equiv 0 \pmod{b_k \prod_{p\in\mathcal{P}(k)} p[(k-1)!]_p}$.

Theorem 2 is proved by using Theorem 1 and a result of Zagier [14]; the following values of $L_2^*(2k-2, f)$ are calculated by the method of [14].

Example 2. Let Δ_k be the normalized eigen cusp form of weight $k=12$, 16, and 20.

(i) $k=12$. $\mathcal{P}(12)=\{23\}$, $b_{12}=691$; $L_2^*(22, \Delta_{12})=(3\cdot 23\cdot 691)^{-1}2^8 7$.

(ii) $k=16$. $\mathcal{P}(16)=\{31\}$, $b_{16}=3617$; $L_2^*(30, \Delta_{16}) =(3\cdot 31\cdot 3617)^{-1}2^9 7^2 11$.

(iii) $k=20$ (Kurokawa [4]). $\mathcal{P}(20)=\phi$, $b_{20}=283\cdot 617$; $L_2^*(38, \Delta_{20}) =(3^2 5\cdot 283\cdot 617)^{-1}2^8 7^2 11\cdot 71^2$.

§3. **Fourier coefficients of** $[f]$. For $F\in S_k(\Gamma_1)$, let $[F]\in M_k(\Gamma_2)$ be the Eisenstein series of degree two attached to F in the sense of Langlands and Klingen, cf. Kurokawa [4], [5] and [6], [8]. For a symmetric positive definite $T=\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}$ with $t_1, t_2, t\in \boldsymbol{Z}$, we put $e(T) =\text{g.c.d.}(t_1, t_2, t)$. Let $f\in S_k(\Gamma_1)$ be a normalized eigen cusp form. Using the explicit formula of $a(T, [f])$ in [8] (Part II) and Theorems 1, 2 above, we have the following.

Theorem 3. *The notation being as in Theorem* 2, *put* $c_k=\text{g.c.d.}(\mathrm{Num}(B_k/k), \mathrm{Num}(B_{2k-2})^\infty)$, $r_k=2^{k-4}\prod_{p\in\mathcal{P}(k)} p\cdot \prod_{p\in\mathcal{P}(k),\, p\mid(k-1)} p$, $s_k =3c_k \prod_{p\notin\mathcal{P}(k),\, p\leqq k-2}[(k-2)!]_p\cdot \prod_{p\notin\mathcal{P}(k),\, p\mid(k-1),\, (p-1)\mid(2k-2)}([k-1]_p/p)$, *and* $t_k =3(k-2)!\,\mathrm{Num}(B_k/k)\prod_{(p-1)\mid(2k-2),\, p\mid(k-1)}([k-1]_p/p)$.

(1) *For each* $T>0$, *put* $A_k(T)=t_k\kappa\nu\,\mathrm{Num}(L_2^*(2k-2, f))\,|e(T)^{-1}2T|^2$. *Then*:

$$a(T, [f])\in \frac{2^{k-4}}{A_k(T)\mathfrak{D}(\boldsymbol{Q}(f))}\boldsymbol{Z}(f).$$

(2) *Assume* $(*)$ *in* §1 *holds. Put* $A_k'(T)=s_k\kappa\,\mathrm{Num}(L_2^*(2k-2, f)) \cdot|e(T)^{-1}2T|^2$. *Then*:

$$a(T, [f])\in \frac{r_k}{A_k'(T)\mathfrak{D}(\boldsymbol{Q}(f))}\boldsymbol{Z}(f).$$

Remark 1. By Kurokawa [5], $\sup_T \mathrm{D}(a(T, [f]))<\infty$. Hence in the definitions of $A_k(T)$ and $A_k'(T)$ above, we may replace $|e(T)^{-1}2T|^2$ by $\text{g.c.d.}(|e(T)^{-1}2T|^2, N_k)$ with some positive integer N_k. It seems that $N_k=\mathrm{D}(B_{2k-2})$ is sufficient.

Corollary. *Under the condition* $(*)$, *suppose a prime* $p\in\mathcal{P}(k)$ *satisfies* $(p, c_k\kappa\,\mathrm{Num}(L_2^*(2k-2, f))\mathfrak{D}(\boldsymbol{Q}(f)))=1$. *Then we have* $a(T, [f]) \equiv 0 \pmod p$ *for all* $T>0$ *such that* $p\nmid |e(T)^{-1}2T|$. *Here the congruences are considered in* $S_p^{-1}\boldsymbol{Z}(f)$, *where* $S_p=\{\alpha\in \boldsymbol{Z}(f) \mid (\alpha, p)=1\}$.

Example 3. Let the notation be as in Example 2.

(i) $k=12$. In this case, $p=23$ satisfies the condition of the corollary. Hence we have: $a(T,[\Delta_{12}])\equiv 0 \pmod{23}$ if $23 \nmid |e(T)^{-1}2T|$. This coincides with Satz 5(a) in Böcherer [1].

(ii) $k=16$. As in (i) we have: $a(T,[\Delta_{16}])\equiv 0 \pmod{31}$ if $31 \nmid |e(T)^{-1}2T|$; e.g., $a((1,1,1),[\Delta_{16}])=(7^2 11)^{-1}2^2 31$, $a((1,1,0),[\Delta_{16}])=(7^2 11)^{-1}2\cdot 3\cdot 29\cdot 31$.

(iii) $k=24$. Let $\Delta_{24}^{(+)}$ be as in Example 1, and $\Delta_{24}^{(-)}=(\Delta_{24}^{(+)})^{\sigma}$ where σ is the non-trivial automorphism of $\boldsymbol{Q}(\sqrt{144169})$. Then as in (i) we have: $a(T,[\Delta_{24}^{(\pm)}])\equiv 0 \pmod{47}$ if $47 \nmid |e(T)^{-1}2T|$. Note that 47 remains prime in $\boldsymbol{Q}(\sqrt{144169})$.

Remark 2. The above congruences have been conjectured in Kurokawa [4] (Part II). We note that $l=23$ (resp. $l=31$, $l=47$) is an exceptional prime for Δ_{12} (resp. Δ_{16}, $\Delta_{24}^{(\pm)}$) of type (ii) in the sense of Serre [11], Swinnerton-Dyer [12], and Ribet [10].

Theorem 4. *Suppose* $(*)$ *in* § 1 *holds. Let the notation be as in Theorem* 3(2) *and let* b_k *be as in Theorem* 2(2). *For* $T_j>0$ $(j=1,2)$ *such that* $e(T_1)=e(T_2)$ *and* $|2T_1|=|2T_2|$, *we have*:

$$a(T_1,[f])-a(T_2,[f])\in \frac{r_k b_k}{A_k'(T_1)\mathfrak{D}(\boldsymbol{Q}(f))}Z(f).$$

Example 4. $a((1,4,1),[\Delta_{12}])-a((2,2,1),[\Delta_{12}])=2^{11}3\cdot 23\cdot 691$ with $b_{12}=691$, and $a((1,4,1),[\Delta_{16}])-a((2,2,1),[\Delta_{16}])=2^{10}3^2 31\cdot 3617$ with $b_{16}=3617$.

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