# 84. $S^1$ Actions with Only Isolated Fixed Points on Almost Complex Manifolds

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**1.** The purpose of this note is to announce some results on $S^1$ actions having only isolated fixed points on an almost complex manifold admitting a quasi-ample complex line bundle. Details will appear elsewhere.

Let $M$ be an oriented, connected, closed $C^\infty$ manifold on which a smooth action of $S^1$ is given such that its fixed points are all isolated. Then the fixed point set consists of exactly $\chi$ points $\{P_i\}$ where $\chi$ denotes the Euler number of $M$. Let $E$ be a complex line bundle over $M$ such that the given $S^1$ action can be lifted to an action on the line bundle $E$. We call such a line bundle admissible. If $E$ is an admissible line bundle, then we fix a lifting of the action on $E$ and consider the fiber $E_{P_i}$ of $E$ over a fixed point $P_i$. $E_{P_i}$ is a complex $S^1$-module so that it can be written in the form

(1.1) $$E_{P_i}=t^{a_i}$$

where $t$ denotes the standard 1 dimensional $S^1$-module. The integer $a_i$ will be called the weight of $E$ at $P_i$. We note that if we choose another lifting of action then the weights $a_i$ are changed simultaneously to $a_i+a$ for some $a$. An admissible line bundle over an even dimensional manifold $M$ will be called quasi-ample if the weights $a_i$ are all different and

$$(c_1(E))^n[M]\neq 0, \qquad \dim M=2n,$$

where $c_1(E)$ denotes the first Chern class of $E$.

Now we assume that $M$ is an almost complex manifold and the action of $S^1$ preserves the almost complex structure. Such a manifold will be called almost complex $S^1$-manifold. Then, restricting the complex tangent bundle $TM$ to each fixed point $P_i$, we get an $S^1$-module

$$TM\,|\,P_i=\sum_k t^{m_{ik}}$$

where the $m_{ik}$ are non-zero integers. These integers $m_{ik}$ are called weights of $M$ at $P_i$. Later we shall consider the following condition (D) relating a quasi-ample line bundle $E$ and the tangent bundle:

(D) There exist integers $k_0\geq 0$ and $d$ such that the identity

$$\sum_k m_{ik}=k_0a_i+d$$

holds for all $i$.

This condition is satisfied if the first Chern classes of $M$ and $E$ are related by the identity $c_1(M)=k_0c_1(E)$.

In the sequel the rational function

$$\varphi_i(t)=\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod_k(1-t^{m_{ik}})} \tag{1.2}$$

will play an important role.

2. Hereafter $M$ will be an almost complex $S^1$-manifold of complex dimension $n$ with only isolated fixed points. Our main results are stated as follows.

**Proposition 2.1.** *The weights $\{m_{ik}\}$ at the fixed points $\{P_i\}$ satisfy the identity*

$$\sum_i \prod_k \left(\frac{1}{1-t^{m_{ik}}}-\lambda\right)=\sum_{q=0}^{n}\rho_q(1-\lambda)^q(-\lambda)^{n-q}$$

*where $\lambda$ is an indeterminate and the integers $\rho_q$ are defined in the following way. For each $i$ we denote by $p_i$ the number of $k$ such that $m_{ik}>0$. $\rho_q$ is defined to be the number of $i$ such that $p_i=q$. Moreover the equality $\rho_{n-q}=\rho_q$ holds for all $q$.*

The above identity shows, in particular, that certain Chern numbers of the almost complex manifold $M$ can be expressed by the integers $\rho_q$. For instance, putting $\lambda=0$, we see that $T[M]=\rho_n=\rho_0$ where $T[M]$ is the Todd genus of the almost complex manifold $M$.

**Corollary 2.2.** *If an almost complex manifold $M$ admits an $S^1$ action having only isolated fixed points, then the Todd genus must be non-negative. Moreover the weights $\{m_{ik}\}$ at the fixed points $\{P_i\}$ satisfy the following relation: For each integer $m$, the number of $(i, k)$ such that $m_{ik}=m$ is equal to the number of $(i, k)$ such that $m_{ik}=-m$. In particular, we have $\sum_{i,k} m_{ik}=0$.*

**Theorem 2.3.** *If there exists a quasi-ample line bundle $E$ over $M$, then the inequality $n+1\leq\chi$ must hold. Moreover, there exist unique elements $r_0(t), \cdots, r_{\chi-1}(t)$ in $\boldsymbol{Z}[t, t^{-1}]$ such that the equalities*

$$\varphi_i(t)=r_0(t)+r_1(t)t^{a_i}+\cdots+r_{\chi-1}(t)t^{(\chi-1)a_i}$$

*hold for all $i$, where the $\varphi_i(t)$ are defined by (1.2) using the weights $a_i$ of $E$ at $P_i$. The function $r_0(t)$ is a constant $r_0$ and we have $r_0=\rho_0=\rho_n=T[X]$.*

**Theorem 2.4.** *Assume, in Theorem 2.3, the quasi-ample line bundle $E$ satisfies the condition (D). Then the inequality $k_0\leq n+1$ holds. Moreover, setting $l_0=\chi-k_0$, the $r_s(t)$ satisfy the relation*

$$r_{l_0-s}(t)=(-1)^{\chi-(n+1)}r_s(t^{-1})t^{-(l_0/\chi)\Sigma a_j}.$$

*In case $k_0=0$ we have $r_0=0$.*

**Corollary 2.5.** *If $c_1(M)=0$, then we have $T[M]=0$.*

The following theorems may be thought of as analogues of Kobayashi-Ochiai's theorem in [2].

**Theorem 2.6.** *Let $E$ be a quasi-ample line bundle over $M$ satisfying the condition* (D). *If $k_0$ is equal to $\chi$, then $k_0=n+1=\chi$ and the weights $\{m_{ik}\}$ at each fixed point $P_i$ are given by*

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i}.$$

*Moreover $M$ is unitary cobordant to the $n$ dimensional complex projective space $CP^n$. In particular we have $T[M]=1$. Furthermore we have $(c_1(E))^n[M]=1$.*

**Remark.** A typical example of Theorem 2.6 is provided by linear $S^1$-actions on $CP^n$. The hyperplane bundle $E$ is quasi-ample and satisfies the condition (D) with $k_0=n+1$.

**Corollary 2.7.** *Suppose that the rational cohomology ring $H^*(M;Q)$ is isomorphic to that of $CP^n$ and*

$$c_1(M)=(n+1)x \text{ mod } \textit{torsion}$$

*for some $x\in H^2(X;Z)$, then the total Chern class of $M$ is of the form*

$$c(M)=(1+x)^{n+1} \text{ mod } \textit{torsion}$$

*and we have $x^n[M]=1$.*

**Theorem 2.8.** *Let $E$ be a quasi-ample line bundle over $M$ satisfying the condition* (D). *If $\chi=n+1$ and $k_0=n$, then $n$ is necessarily odd, and to each fixed point $P_i$ there corresponds another fixed point $P_{i'}$ such that $a_i+a_{i'}=0$ when the $a_i$ are normalized to fulfill*

$$\sum_{i=1}^{n+1} a_j=0.$$

*Moreover the weights $\{m_{ik}\}$ at $P_i$ are given by*

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i,i'}\cup\{a_i\}.$$

*Furthermore $M$ is unitary cobordant to the complex quadric $Q_n$. In particular we have $T[M]=1$ and $(c_1(E))^n[M]=2$.*

**Corollary 2.9.** *Suppose that the rational cohomology ring $H^*(M;Q)$ is isomorphic to that of $CP^n$ and*

$$c_1(M)=nx \text{ mod } \textit{torsion}$$

*for some $x\in H^2(M;Z)$, then $n$ is necessarily odd and the total Chern class of $M$ is of the form*

$$c(M)=(1+x)^{n+2}(1+2x)^{-1}$$

*and we have $x^n[M]=2$.*

**Remark.** A typical example of Theorem 2.8 is provided by linear $S^1$-actions on $Q_n$ for odd $n$. Let $E$ be a quasi-ample line bundle over $M$ satisfying the condition (D) with $k_0=n$ and $\chi(M)=n+2$. With some additional condition we can determine the weights $\{m_{ik}\}$ which are completely similar to linear $S^1$-actions on $Q_n$ for even $n$.

From Theorems 2.3 and 2.4 we have the inequalities $k_0\leq n+1\leq\chi$ for a quasi-ample line bundle $E$ over an almost complex $S^1$-manifold of complex dimension $n$ having only isolated fixed points. Theorem 2.7 completely determines the weights at each fixed point in the most extreme case $k_0=n+1=\chi$. It is natural to ask what happens in case

$k_0=n+1$ or $\chi=n+1$. In view of Kobayashi-Ochiai's theorem it might be conjectured that the same conclusion of Theorem 2.7 holds under a weaker assumption $k_0=n+1$. However, as actions on the Hirzebruch surfaces show, quasi-ampleness is a notion strictly weaker than ampleness. Thus the conjecture above would be a harzardous one. As to the other case $\chi=n+1$ we propose the following conjecture. Let $M$ be an almost complex $S^1$-manifold of complex dimension $n$ having only isolated fixed points such that $\chi(M)=n+1$ and $T[M]\neq 0$. Let $E$ be a quasi-ample line bundle over $M$ satisfying the condition (D).

**Conjecture 2.10.** *If* $(c_1(E))^n[M]=1$ *then* $k_0=n+1$.

**Theorem 2.11.** *Conjecture* 2.10 *is true for* $n\leq 4$.

From Theorem 2.11 and Corollary 2.7 we deduce the following

**Corollary 2.12.** *Suppose that $M$ has the same integral cohomology ring as $\boldsymbol{CP}^n$ and that $T[M]\neq 0$. If $4\leq n$, then the total Chern class of $M$ is of the form*

$$c(M)=(1+x)^{n+1}$$

*where $x$ is a generator of $H^2(M;Z)$ such that $x^n[M]=1$.*

We conjecture that the conclusion of Corollary 2.12 holds for all $n$. T. Petrie [3] conjectured that if $S^1$ acts smoothly with only isolated fixed points on $2n$ dimensional closed manifold having the same cohomology ring as $\boldsymbol{CP}^n$ then the total Pontrjagin class of $M$ was of the form

$$p(M)=(1+x^2)^{n+1}.$$

The conjecture just stated above may be regarded as a complex version of Petrie's conjecture. Related materials can be found in [1].

## References

[1] A. Hattori: Spin$^c$-structures and $S^1$-actions. Invent. math., **48**, 7–31 (1978).
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