81. *The Thue-Siegel-Roth Theorem for Values of Algebraic Functions*

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§ 0. The Thue-Siegel-Roth theorem [1], [2] on the approximation of algebraic numbers by rational numbers states that for a given algebraic number α and any $\varepsilon>0$ there exist only finitely many rational approximations p/q to α such that $|\alpha-p/q|<|q|^{-2-\varepsilon}$. Here, as well as in earlier versions [3], the results are ineffective; they depend on the knowledge of at least one good approximation to α. Such good approximations are known only for an α of a special form, see [4] and [5]. Here we prove an entirely new theorem on approximations of algebraic numbers by considering diophantine approximations to values of algebraic functions. Our result shows that for an algebraic functions defined over $\boldsymbol{Q}(x)$ and regular at $x=0$, the "Roth" theorem holds for the number $f(r)$ with a rational $r\neq 0$ close to 0. Our methods are based on the Wronskian technique developed in [6] for the functional version of Roth's theorem. A complete proof is presented for cubic algebraic functions, satisfying Ricatti equations.

§ 1. Let $f(x)$ be an algebraic function over $\boldsymbol{Q}(x)$ defined as a solution of an algebraic equation $P(x, f(x))=0$ for an absolutely irreducible polynomial $P(x, y)$ over $\boldsymbol{Q}[x, y]$. We also assume that $f(x)$ is regular at $x=0$ and has the Taylor expansion $f(x)=\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \boldsymbol{Q}$.

Theorem 1. *Let $f(x)$ be as above and let $r=a/b$ for rational integers a and b. For every $\varepsilon>0$ there exist effective constants $c_1=c_1(\varepsilon, f)>0$ and $c_2=c_2(\varepsilon, a, b, f)$ with the following properties. If $|b|^{\varepsilon}\geq c_1\cdot|a|^{2(1+\varepsilon)}$, then*

$$\left|f(r)-\frac{P}{Q}\right|>|Q|^{-2-\varepsilon}$$

for arbitrary relatively prime rational integers P, Q with $|Q|\geq c_2$.

A similar result holds in the p-adic metric, if one replaces $|b|$ with $|b|_p$ and $|f(r)-P/Q|$ by $|f(r)-P/Q|_p$. The proof of this theorem is based on the author's studies of generalizations of Padé approximations to solutions of linear differential equations and their relation to the Ricatti equation and Wronskian calculus [6]. Theorem 1 above holds for an arbitrary (G, C)-function $f(x)$ (see the definition in [7]). Moreover, these results can be generalized for the simultaneous diophantine approximations of values of several (G, C)-functions.

Theorem 2. *Let $f_1(x), \cdots, f_n(x)$ be n functions algebraic over $\boldsymbol{Q}(x)$ that have the Taylor series expansions at $x=0$ with rational coefficients. Let $\varepsilon>0$ and $r=a/b$ for rational integers a and b. If $|b|^{\varepsilon}>c_3\cdot|a|^{n(n-1+\varepsilon)}$, then*

$$|H_1f_1(r)+\cdots+H_nf_n(r)|>H^{-n+1-\varepsilon}$$

for $H=\max(|H_1|, \cdots, |H_n|)$, provided that $H\geq c_4$ and $H_1f_1(r)+\cdots+H_nf_n(r)\neq 0$. Here $c_3=c_3(\varepsilon, f_1, \cdots, f_n)>0$ and $c_4=c_4(\varepsilon, a, b, f_1, \cdots, f_n)$ are effective constants.

Modifications in the proofs of Theorems 1 and 2 allow us to obtain sharp effective results under much less restrictive conditions on r.

Remark 3. If $r=a/b\neq 0$ and $|r|<1$, then under the assumptions of Theorem 1, we have $|f(r)-P/Q|>|Q|^{\chi-\varepsilon}$ for $|b|\geq c_1'$, $|Q|\geq c_2'$ and $\chi=2\log|b|/\log|a/b|$.

The Thue theorem follows from Theorem 1, but Theorem 1 applies to new classes of algebraic numbers. Among them are, in particular, classes of algebraic numbers that are roots of polynomials arising as a non-singular one-parametric deformations of a given polynomial $P(x)\in \boldsymbol{Z}[x]$.

Example. Let $\alpha=R(N)^{m/n}$ for a rational function $R(x)\in\boldsymbol{Q}(x)$ such that $R(x)\to 1$ as $x\to\infty$. Then α satisfies "Roth's $2+\varepsilon$ theorem" for an integer $N\geq N_0(\varepsilon, n)$: $|\alpha-P/Q|>|Q|^{-2-\varepsilon}$ for $|Q|\geq Q_1(\varepsilon, N, n)$. We want to remark that in this case the function $f(x)=R(x)^{m/n}$ has the Taylor expansion at $x=0$ with rational coefficients, and satisfies a linear differential equation of the first order.

§2. We present a complete proof of Theorem 1 in the case of $f(x)$ satisfying a Ricatti equation over $\boldsymbol{Q}(x)$. This is, in particular, the case of an arbitrary cubic algebraic function $f(x)$, because for cubic irrationality f, the functions f', f^2, f, 1 are linearly dependent over $\boldsymbol{Q}(x)$ [8]. Ricatti equation is also satisfied in the case of the function $f(x)$ from the example above.

For the proof of Theorem 1 we construct Padé-type approximations to $f(x)$ at $x=0$. Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$, $a_n\in\boldsymbol{Q}$. According to the Eisenstein theorem, there exists an integer A (depending only on $P(x, y)$) such that all numbers $A^n\cdot a_n$ are integers. (If, $P_y(0, f(0))\neq 0$, then $A\,|\,(P_y(0, f(0)))^2$.) We fix $\delta>0$ and take a sufficiently large integer N. Then, according to Dirichlet's box principle there exists a nonzero polynomial $Q(x)\in\boldsymbol{Z}[x]$ of degree of at most N of height of at most $\exp\{C_1N/\delta\}$ such that the following condition is satisfied. If $P(x)$ denotes the sum of the first n terms of the Taylor expansion of $Q(x)f(x)$ at $x=0$, then $\mathrm{ord}_{x=0}(Q(x)f(x)-P(x))\geq(2-\delta)\cdot N$.

Let an irreducible Ricatti equation satisfied by $f(x)$ be $f'=cf^2$

$+df+e$, for $c, d, e \in \boldsymbol{Q}(x)$. We denote the remainder function $Q(x)f(x)-P(x)$ by $R(x)$. Then for an arbitrary $k \geq 0$, $(d/dx-cf)^k R(x) = Q_k(x)f(x)-P_k(x) \overset{\text{def}}{=} R_k(x)$, where (rational functions) $Q_k(x)$, $P_k(x)$ are defined recursively as $Q_{k+1}=Q_k'+dQ_k+cP_k$, $P_{k+1}=P_k'-eQ_k$.

The determinant $W(x)=\begin{vmatrix} Q_1 & P_1 \\ Q & P \end{vmatrix}$ is identically zero, if and only if the approximation $r(x)=P(x)/Q(x)$ satisfies the Ricatti equation $r' = cr^2+dr+e$. This is impossible when the Ricatti equation is irreducible, i.e. does not have any rational solution. The derivatives $(d/dx)^s w(x)$ of $w(x)$ can be expressed as linear combinations of determinants $\begin{vmatrix} Q_k & P_k \\ Q_l & P_l \end{vmatrix}$ for $k, l \leq s+1$. Let us take a rational number $r=a/b \neq 0$. According to the definition, $W(x)$ is a rational function $Q^2 \cdot \{(P/Q)' - c(P/Q)^2 - d(P/Q) - e\}$ and thus the degree $\deg W(x)$ does not exceed $2 \cdot \max\{\deg(P), \deg(Q)\} + \deg(c) + \deg(d) + \deg(e)$. On the other hand, $W(x) = \begin{vmatrix} Q_1 & R_1 \\ Q & R \end{vmatrix}$ so that $\operatorname{ord}_{x=0} W(x) \geq (2-\delta)N-1$.

Hence, we have $\operatorname{ord}_{x=r} W(x) \leq \delta N + C_2$. This means, according to the description of $(d/dx)^s W(x)$ above, that among the vectors $(Q_k(r), P_k(r))$: $k=0, \cdots, \delta N+C_2+1$ there are at least two that are linearly independent. This implies that for an arbitrary A (e.g, $A=f(r)$ or $A=p/q$ for an approximation p/q to $f(r)$), there exists a $k \leq \delta N + C_2 + 1$ such that $Q_k(r)A-P_k(r) \neq 0$.

Next, $R(x)=\sum_{m \geq [(2-\delta)N]} b_m x^m$, where $b_m=\sum_{i=0}^{N} q_i a_{m-i}$ and $Q(x) = \sum_{i=0}^{n} q_i x^i$. This implies that for r within the radius of the convergence of $f(x)$, $|1/k!\, R_k(r)| \leq |r|^{(2-\delta)N-k} \exp\{C_3 k + C_3 N/\delta\}$. Let p/q be a rational approximation to $f(r)$, $|f(r)-p/q| < |q|^{-\mu}$ for $\mu > 2$. We choose the smallest $k \leq \delta N + C_2 + 1$ such that $Q_k(r)p/q - P_k(r) \neq 0$. According to the definition of $R_k(x)$, the denominators of $1/k!\, P_k(x)$, $1/k!\, Q_k(x)$ divide $D(x)^k$ for a fixed polynomial $D(x) \in \boldsymbol{Z}[x]$ depending only on c, d, e. Hence $1/k!\, P_k(r)D(r)^k b^{N_k}$ and $1/k!\, Q_k(r)D(r)^k b^{N_k}$ are rational integers, with $N_k = \max\{\deg(Q_k D^k), \deg(P_k D^k)\} \leq N + C_4 k$. Then

$$1 \leq |1/k!\, D(r)^k \cdot b^{N_k}| \cdot |pQ_k(r) - qP_k(r)| \leq |D(r)^k \cdot b^{N_k} \cdot q| \cdot |1/k!\, R_k(r)| + |D(r)^k \cdot b^{N_k} \cdot q| \cdot |1/k!\, Q_k(r)| \cdot |q|^{-\mu}.$$

This implies $1 \leq |q| \cdot |a|^{(2-2\delta)N} \cdot |b|^{(-1+2\delta)N + C_5 \delta N} \cdot \exp\{C_6 N/\delta\} + |q|^{1-\mu} \cdot |b|^{C_5 \delta N + N} \cdot \exp\{C_7 N/\delta\}$. By choosing an appropriate N, say

$$N = [-\mu \log|q| / \{(2-2\delta)\log|a/b| + C_8/\delta\}]$$

with a sufficiently small δ, we obtain an upper bound on μ. Taking into account the trivial (Liouville) bound on μ, we obtain

$$\mu \leq \frac{2 \cdot \log|a/b|}{\log|a^2/b|} + \varepsilon,$$

provided that $|b| > |a|^2$ and $|b| \geq B_0(\varepsilon, f)$. In particular, Theorem 1 is proved.

The constants C_i in the proof can be effectively determined in terms of the Ricatti equation satisfied by $f(x)$. Moreover, in some interesting cases, one can explicitly determine Padé-type approximations $R(x)$ to $f(x)$ from the proof, in the case $\delta=0$. This provides exceptionally good bounds for the measure of irrationality of $f(a/b)$ for small $|a/b|$. This is the case of $f(x)=\sqrt[n]{1+x}$ and some classes of cubic irrationalities studied in [5]. We present an entirely new class of cubic irrationalities, where explicit Padé approximations can be determined. This is the case of $f^3+f/x-1=0$, where $f\sim x$ as $x\to 0$. The Padé approximation to this branch of $f(x)$ can be expressed in terms of hypergeometric functions, with a remainder function of the form $R(x)=x^{2n+1}\cdot {}_2F_1(7/6+n, 5/6+n; 5/2+2n; -27x^3/4)$ for $n=0, 1, \cdots$. This implies the effective measure of irrationality of the smallest root α of $\alpha^3+a\alpha-1=0$, $a\in \boldsymbol{Z}$, as $|\alpha-p/q|>|q|^{\chi-1-\varepsilon}$ for $|q|\geq q_1(a,\varepsilon)$ with $\chi=(\pi+\sqrt{3}\log|w|)/(\pi-\sqrt{3}\log|w|)$, $w=16a^3/27$, $|a|\geq 3$. For small $|a|$ see [5, Ch. VIII].

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