# 80. On Degrees of Non-Roughness of Real Projective Varieties

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Related to *the Hilbert's 16th problem*, the following problem is presented since 1965 (see Gudkov [3] p. 485, [4] p. 6 and Risler [6] p. 23):

**Problem.** *Does each real plane algebraic curve of a fixed order have a well-defined and finite degree of non-roughness?*

In short, the degree of non-roughness of a curve (or a variety) represents its topological degeneration (cf. Definition 2).

The purpose of the present note is to answer this problem *affiermatively* thanks to the stratification theory of R. Thom ([8]). Further we see that *the degrees of non-roughness of real projective varieties of a fixed order are well-defined and have a finite upper bound* (Theorem 1).

We consider the "equivariant" isotopy type of a complexified variety: Theorem 2 (cf. [7]).

**1. Formulations of results.** Let $\boldsymbol{RP}^{N_1}\times\cdots\times\boldsymbol{RP}^{N_s}$ be the set of $f=(f_1,\cdots,f_s)$ considered modulo non-zero-constants in each component, where $f_i$ is a non-zero homogeneous polynomial of order $d_i$, with variables $x_0, x_1, \cdots, x_n$ and with coefficients in $\boldsymbol{R}$, and $N_i=\binom{n+d_i}{n}-1$ $(i=1,\cdots,s)$.

We mean by a *real projective variety* of order $(d_1,\cdots,d_s)$ simply a point of $\boldsymbol{RP}^{N_1}\times\cdots\times\boldsymbol{RP}^{N_s}$. Each real projective variety $[f]$ determines naturally a subset $V[f]$ of $\boldsymbol{RP}^n$ and invariant subset $\boldsymbol{C}V[f]$ of $\boldsymbol{CP}^n$ under the complex conjugation, by the equation $f_1(x)=\cdots=f_s(x)=0$.

The first half of the sixteenth problem of Hilbert is regarded, in an extended sense, as the investigation of isotopy types of pairs $(\boldsymbol{RP}^n, V[f])$ (cf. [4]).

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a semi-algebraic stratification of a closed subset $A$ of $\boldsymbol{RP}^n$ (resp. $B$ of $\boldsymbol{CP}^n$, $\mathcal{B}$ being invariant under the complex conjugation $\boldsymbol{CP}^n\to\boldsymbol{CP}^n$). (A subset of an algebraic manifold is *semi-algebraic* if it is semi-algebraic on each affine chart.)

**Definition 1.** Two real projective varieties $[f], [f']\in\boldsymbol{RP}^{N_1}\times\cdots\times\boldsymbol{RP}^{N_s}$ of a same degree $(d_1,\cdots,d_s)$ are called *isotopic rel.* $\mathcal{A}$ (resp.

*equivariantly isotopic rel.* $\mathcal{B}$) if there exists a continuous one parameter family of homeomorphisms $h_t: \boldsymbol{RP}^n \to \boldsymbol{RP}^n$ (resp. $h_t: \boldsymbol{CP}^n \to \boldsymbol{CP}^n$ commuting with the complex conjugation), $t \in [0, 1]$, such that $h_0$ is the identity, each $h_t$ maps each stratum of $\mathcal{A}$ (resp. $\mathcal{B}$) to itself, and

$$h_1(V[f] \cap A) = V[f'] \cap A \qquad (\text{resp. } h_1(\boldsymbol{C}V[f] \cap B) = \boldsymbol{C}V[f'] \cap B).$$

The following definition is based on that due to Andronov and Gudkov [3], [4] (with the origin in [1]).

**Definition 2.** A real projective variety $[f] \in X = \boldsymbol{RP}^{N_1} \times \cdots \times \boldsymbol{RP}^{N_s}$ is *of degree of non-roughness* 0 (alternatively, *rough, rigid, stable, grossier*) if there exists a neighborhood $U$ of $[f]$ in $X$ such that any $[f'] \in U$ is isotopic to $[f]$ rel. $\mathcal{A}$. Inductively, a real projective variety $[f]$ is *of degree of non-roughness* $r+1$ if, firstly, for any neighborhood $V$ of $[f]$, there exists a $[g] \in V$ of degree of non-roughness $r$ which is not isotopic to $[f]$ rel. $\mathcal{A}$, and, secondly, there exists a neighborhood $W$ of $[f]$ such that any $[h] \in W$ that is not of degree of non-roughness $k \leq r$ is isotopic to $[f]$ rel. $\mathcal{A}$.

**Theorem 1.** *Let $\mathcal{A}$ be as above. Then the degrees of non-roughness of real projective varieties are well-defined and have a finite upper bound depending on the number of variables and the order.*

If we put $s=1$ and $n=2$, then our Theorem 1 and Remark 1 below answer the previous problem:

**Corollary.** *Each real plane algebraic curve of a fixed order has a well-defined and finite degree of non-roughness.*

The notion of degree of non-roughness (Definition 2) can be extended, word for word, to the case of an arbitrary topological space $X$ with an equivalence relation $E$ (cf. [3], § 8).

**Theorem 2.** *Let $\mathcal{B}$ be as above. Then the degree of non-roughness of real projective varieties with respect to the equivalence of equivariant isotopy rel. $\mathcal{B}$ are well-defined and have a finite upper bound depending on the number of variables and the order.*

**2. General properties of degrees of non-roughness.** Let $X$ be a topological space and $E$ be an equivalence relation on $X$. We consider the degrees of non-roughness of points in $X$ (see § 1).

**Lemma 1.** *If a point $x \in X$ has a finite degree of non-roughness, then it is uniquely determined.*

**Lemma 2.** *A point $x \in X$ is of degree of non-roughness $r+1$ if and only if $x$ is an interior point of the $E$-equivalence class of itself in the complement to $X$ of the subspace of points of degrees of non-roughness $\leq r$ $(r=-1, 0, 1, \cdots)$.*

**Remark 1.** In the papers of Gudkov [3], [4], the degrees of non-roughness are defined on a subspace of the totality of real plane algebraic curves, and the equivalence of isotopy considered in them is slightly weaker than ours.

In general, the statement that each point in a subset $Y \subseteq X$ has a finite (resp. bounded) degree of non-roughness with respect to $(Y, E')$ follows from the same statement for $(X, E)$, $E$ being finer than $E'$.

**3. Outlines of proofs of theorems.** Recall that $X = \boldsymbol{RP}^{N_1} \times \cdots \times \boldsymbol{RP}^{N_s}$ is the space of real projective varieties of order $(d_1, \cdots, d_s)$ with $(n+1)$-homogeneous variables. We put

$$V = \{([x], [f]) \in \boldsymbol{RP}^n \times X \mid f_1(x) = \cdots = f_s(x) = 0\}$$

and denote $\pi : \boldsymbol{RP}^n \times X \to X$ the projection to the latter factor. The fiber $\pi^{-1}[f] \cap V$ of $\pi | V$ is just $V[f]$.

In the situation on Theorem 1, we denote $\mathcal{A}^+$ the product stratification of $\mathcal{A}$ and the trivial stratification of $X$. As well-known in the stratification theory (cf. [8], III C), we can construct semi-algebraic Whitney stratifications $S$ and $T$ of $\boldsymbol{RP}^n \times X$ and $X$ respectively such that $V$ (resp. each stratum of $\mathcal{A}^+$) is a union of strata of $S$ and $\pi$ is a stratified mapping with respect to $(S, T)$.

For each stratum $Z \in T$, $\pi^{-1}(Z)$ is a union of strata of $S$, and the family $\pi : (\pi^{-1}(Z), S | \pi^{-1}(Z)) \to Z$ is locally topologically trivial by Thom's first isotopy lemma (cf. [2]).

Let $E_1$ (resp. $E_2$) be the equivalence relation on $X$ of isotopy rel. $\mathcal{A}$ (resp. induced by the decomposition $T^\circ$ by connected components of strata of $T$). Then we see that $E_2$ is finer than $E_1$.

The degree of non-roughness of a real projective variety $[f] \in X$ with respect to $E_2$ equals to the $T$-depth of $[f]$ (cf. [5]) and we have an upper bound (at most $\dim X = N_1 + N_2 + \cdots + N_s$).

By Remark 1, the degrees of non-roughness with respect to $E_1$ have also an upper bound.

Lemma 1 and this imply Theorem 1.

**Remark 2.** The mapping $\pi | V : V \to X$ does not admit any *Thom stratification* except for the case $s=1$, $d_1=1$ or $n=1$.

For the proof of Theorem 2, we put $\boldsymbol{CV} = \{([x], [f]) \in \boldsymbol{CP}^n \times X \mid f_1(x) = \cdots = f_s(x) = 0\}$ and consider the involution $\mathrm{conj} \times 1_X$ on $\boldsymbol{CP}^n \times X$. Then we can construct invariant semi-algebraic Whitney stratifications $S'$ and $T'$ of $\boldsymbol{CP}^n \times X$ and $X$ respectively with properties as those of $S$ and $T$, and we apply the equivariant isotopy lemma, which can be proved similarly to the non-equivariant case.

**Remark 3.** Theorem 1 (resp. Theorem 2) is also valid in the case $\mathcal{A}$ (resp. $\mathcal{B}$) is subanalytic.

## References

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