77. *Three Commodity Flows in Graphs*

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(Communicated by Shokichi IYANAGA, M. J. A., June 14, 1983)

Let $G=(V,E)$ be a graph (finite undirected, possibly with multiple edges but without loops). In this paper a path has no repeated edges, and we permit the path with one vertex and no edges. For two distinct vertices x, y we let $\lambda(x, y)=\lambda_G(x, y)$ be the maximal number of edge-disjoint paths between x and y, and we let $\lambda(x, x)=\infty$.

We first consider the following problem.

Let $(s_1, t_1), \cdots, (s_k, t_k)$ be pairs (not necessarily distinct) of vertices of G. When is the following true?

(1.1) There exist edge-disjoint paths $P_1, \cdots, P_k$ such that P_i has ends s_i, t_i $(1\leq i\leq k)$.

Seymour [9] and Thomassen [11] answered to this problem when $k=2$, and Seymour [9] when $s_1, \cdots, s_k, t_1, \cdots, t_k$ take only three distinct values.

Our result is the following

Theorem 1. *Suppose that $s_1, s_2, s_3, t_1, t_2, t_3$ are vertices of a graph G. If for each $i=1, 2, 3$ $\lambda(s_i, t_i)\geq 3$, then there exist edge-disjoint paths P_1, P_2, P_3 of G, such that P_i has ends s_i and t_i $(i=1, 2, 3)$.*

If $\lambda(s_i, t_i)\leq 2$ for some i, then this conclusion does not hold. Fig. 1 gives a counterexample.

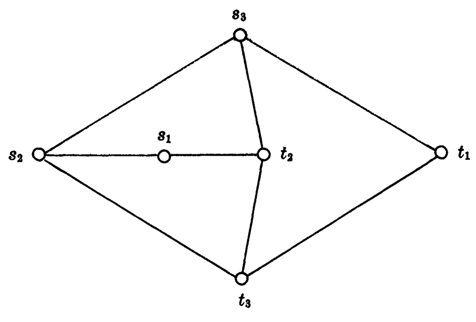

Fig. 1

For a positive integer k, we let $g(k)$ be the smallest integer such that for every $g(k)$-edge-connected graph and for every vertices $s_1, \cdots, s_k, t_1, \cdots, t_k$ of the graph, (1.1) holds. Thomassen [11] conjectured the following.

Conjecture (Thomassen). *For each odd integer $k\geq 1$, $g(k)=k$, and for each even integer $k\geq 2$, $g(k)=k+1$.*

If k is even then $g(k)>k$ (see [11]). It follows easily from Menger's theorem that $g(k)\leq 2k-1$, thus $g(1)=1$, $g(2)=3$; and Cypher [1] proved $g(4)\leq 6$ and $g(5)\leq 7$. As a corollary of Theorem 1 we have the following.

Corollary. $g(3)=3$.

The second problem we consider is the multicommodity flow problem.

Suppose that each edge $e\in E$ has a real-valued capacity $w(e)\geq 0$, and each path has a positive value. We assume that $w\equiv 1$ and each path has value 1 when there is no explanation. For a positive number α, path αP, P denote paths of value α, 1 respectively. We say that a set of paths $\alpha_1 P_1, \cdots, \alpha_n P_n$ is feasible if for each edge $e\in E$,

$$\sum_{i\in\{i\mid e\in E(P_i)\}} \alpha_i \leq w(e),$$

where $E(P_i)$ is the set of edges of P_i.

For two vertices x, y and a real number $q>0$, a flow F of value q between x and y is a set of paths $\alpha_1 P_1, \cdots, \alpha_n P_n$ between x and y such that $\alpha_1+\cdots+\alpha_n=q$. When $\alpha_1, \cdots, \alpha_n$ are all integers (half-integers), F is called an integer (half-integer) flow. We say that a set of flows $F_1, \cdots, F_k$ is feasible if the set of paths of $F_1, \cdots, F_k$ is feasible.

Now the *multicommodity flow problem* is formulated as follows.

Let $(s_1, t_1), \cdots, (s_k, t_k)$ be pairs of vertices of G, as before, and suppose that $q_i\geq 0$ $(1\leq i\leq k)$ are real-valued demands. When is the following true?

(1.2) There exist feasible flows $F_1, \cdots, F_k$, such that F_i has ends s_i and t_i and value q_i $(1\leq i\leq k)$.

Remark. When $k=3$, $w\equiv 1$, and $q_i=1$ $(1\leq i\leq 3)$, Theorem 1 implies that (1.2) is true if $\lambda(s_i, t_i)\geq 3$ $(1\leq i\leq 3)$, and then the flows may be chosen as integer flows.

For a set $X\subseteq V$, we let $\partial(X)=\partial_G(X)\subseteq E$ be the set of edges with one end in X and the other in $V-X$, and we let $D(X)\subseteq\{1, 2, \cdots, k\}$ be

$$\{i\mid 1\leq i\leq k,\ X\cap\{s_i, t_i\}\neq\phi\neq(V-X)\cap\{s_i, t_i\}\}.$$

It is clear that if (1.2) is true, then the following holds.

(1.3) For each $X\subseteq V$,

$$\sum_{e\in\partial(X)} w(e)\geq \sum_{i\in D(X)} q_i.$$

Note that $\sum_{e\in\partial(x)} w(e)=|\partial(X)|$ if $w\equiv 1$, and $\sum_{i\in D(X)} q_i=|D(X)|$ if $q_i=1$ for any i.

Our second result is the following

Theorem 2. *Suppose that G is a graph and w is integer-valued, and that $k=3$, $q_1=q_2=q_3=1$. Then* (1.2) *and* (1.3) *are equivalent.*

Moreover if (1.3) holds, then the flows F_i in (1.2) may be chosen as half-integer flows.

(1.4) In general (1.2) and (1.3) are not equivalent, but in the following cases they are equivalent.

(1.4.1) $k=1$ (Ford and Fulkerson [2]).

(1.4.2) $k=2$ (Hu [3] and Seymour [7]).

(1.4.3) $k=5$, $t_i=s_{i+1}$ $(i=1, 2, 3, 4)$ and $t_5=s_1$ (Papernov [6]).

(1.4.4) $k=6$, and the (s_i, t_i) correspond to the six pairs of a set of four vertices (Seymour [8] and Papernov [6]).

(1.4.5) $s_1=s_2=\cdots=s_j$ and $s_{j+1}=\cdots=s_k$ (obvious extension of (1.4.2)).

(1.4.6) The graph $(V, E\cup\{e_1, \cdots, e_k\})$ is planar, where the edge e_i has ends s_i and t_i $(1\leq i\leq k)$ (Seymour [10]).

(1.4.7) G is planar and can be drawn in the plane so that $s_1, \cdots, s_k, t_1, \cdots, t_k$ are all on the boundary of the infinite face (Okamura and Seymour [4]).

(1.4.8) G is planar and can be drawn in the plane so that $s_1, \cdots, s_j, t_1, \cdots, t_j$ are all on the boundary of a face and $s_{j+1}, \cdots, s_k, t_{j+1}, \cdots, t_k$ are all on the boundary of the infinite face (Okamura [5]).

(1.4.9) G is planar and can be drawn in the plane so that $s_{j+1}, \cdots, s_k, t_1, t_2, \cdots, t_k$ are all on the boundary of the infinite face, and $t_1=\cdots=t_j$ (Okamura [5]).

Moreover if (1.3) and the following (1.5) hold in each case except (1.4.3), or if (1.3) holds and w, q_i are even-integer-valued in the case (1.4.3), then the flows F_i of (1.2) may be chosen as integer flows.

(1.5) w and q_i are integer-valued, and for each vertex $x\in V$,

$$\sum_{e\in\partial(x)} w(e)-\sum_{i\in D(x)} q_i$$

is even.

(1.4.1)–(1.4.5) are all configurations of (s_i, t_i) for which (1.2) and (1.3) are equivalent for all graphs G and all w, q_i (see [8]). When $q_i>0$ $(1\leq i\leq 3)$, the case of Theorem 2 is the only case for which (1.2) and (1.3) are equivalent for all graphs G and all w, (s_i, t_i). Fig. 1 gives a counterexample with $q_1=2$, $q_2=q_3=1$.

The detailed proofs of the theorems will be published elsewhere.

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