66. *On the Existence of Solution to Schwinger's Functional Differential Equations of Higher Order*

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1. Introduction. In quantum field theory, functional differential equations of special type, called Schwinger equations, appear. Each Schwinger equation corresponds to a model of quantum fields in a formal manner and it is regarded as an equation that contains all the physics of the model in some way. For some examples of Schwinger equations and related, non-rigorous, formal discussions, see, e.g., Rzewski [5].

So far, few rigorous mathematical analyses for Schwinger equations have been made (see, however, Gelfand [2]). Recently, Inoue [4] constructed an explicit solution to a certain Schwinger equation of the first order. Schwinger equations of the first order, however, are not so difficult to handle and we can obtain a general form of solutions to them [1]. In this short note, we report a result on the existence of solution to a certain Schwinger equation of *higher order*.

2. Definitions. We first begin with a precise definition of *functional derivative*. Let $\mathcal{S}(\boldsymbol{R}^n)$, $n=1, 2, 3, \cdots$, be the Schwartz space of rapidly decreasing C^∞ functions. For $T \in \mathcal{S}'(\boldsymbol{R}^n)$ and $f \in \mathcal{S}(\boldsymbol{R}^n)$ we define $(T, f) \in \boldsymbol{C}$ by $(T, f)=T(f)$.

Definition 2.1. Let $Z=Z(f)$ be a complex-valued functional on $\mathcal{S}(\boldsymbol{R}^n)$. If, at the point f in $\mathcal{S}(\boldsymbol{R}^n)$, there exists $Z_1(f) \in \mathcal{S}'(\boldsymbol{R}^n)$ such that for all h in $\mathcal{S}(\boldsymbol{R}^n)$

$$\lim_{\varepsilon\to 0} \frac{Z(f+\varepsilon h)-Z(f)}{\varepsilon} \equiv DZ(h)(f)=(Z_1(f), h), \tag{2.1}$$

then Z is said to be differentiable at the point f and $Z_1(f)$ is called the functional derivative of Z at the point f; we shall denote its distribution kernel by $\delta Z(f)/\delta f(x)$:

$$DZ(h)(f)=\int dx \frac{\delta Z(f)}{\delta f(x)} h(x). \tag{2.2}$$

If Z is differentiable at all points in $\mathcal{S}(\boldsymbol{R}^n)$, then it is said to be differentiable on $\mathcal{S}(\boldsymbol{R}^n)$.

To define functional derivatives of higher order, we note that, if Z is differentiable on $\mathcal{S}(\boldsymbol{R}^n)$, then $DZ(h)$ is also a functional on $\mathcal{S}(\boldsymbol{R}^n)$ for each fixed h.

Definition 2.2. Let Z be a differentiable functional on $\mathcal{S}(\boldsymbol{R}^n)$. If $DZ(h)$ is differentiable on $\mathcal{S}(\boldsymbol{R}^n)$ for each h in $\mathcal{S}(\boldsymbol{R}^n)$ and its functional derivative is continuous linear with respect to h, then Z is said to be two times differentiable on $\mathcal{S}(\boldsymbol{R}^n)$.

If Z is two times differentiable on $\mathcal{S}(\boldsymbol{R}^n)$, then, by the Schwartz nuclear theorem, there exists a unique $Z_2(f) \in \mathcal{S}'(\boldsymbol{R}^{2n})$ such that for all h_1, h_2 and f in $\mathcal{S}(\boldsymbol{R}^n)$

$$D(DZ(h_1))(h_2)(f) = (Z_2(f), h_1 \otimes h_2) = (Z_2(f), h_2 \otimes h_1), \tag{2.3}$$

where $h_1 \otimes h_2 \in \mathcal{S}(\boldsymbol{R}^{2n})$ is defined by

$$(h_1 \otimes h_2)(x, y) = h_1(x) h_2(y), \qquad x, y \in \boldsymbol{R}^n. \tag{2.4}$$

The continuous linear functional $Z_2(f)$ on $\mathcal{S}(\boldsymbol{R}^{2n})$ is called the functional derivative of the second order of Z at the point f; we shall denote its distribution kernel by $\delta^2 Z(f)/\delta f(x)\delta f(y)$, which, by (2.3), is symmetric with respect to x and y. In the same way we can define successively the functional derivative of the m-th order of Z at each point in $\mathcal{S}(\boldsymbol{R}^n)$ as an element in $\mathcal{S}'(\boldsymbol{R}^{mn})$; we shall denote its distribution kernel by $\delta^m Z(f)/\delta f(x_1)\cdots\delta f(x_m)$, which is symmetric with respect to all permutations of $\{1, \cdots, m\}$.

In order to define Schwinger equations of higher order, we must introduce a concept of *re-ordering of functional derivatives*:

Definition 2.3. Let $Z = Z(f)$ be a m-times differentiable functional on $\mathcal{S}(\boldsymbol{R}^n)$ and $C(x, y)$ be a locally integrable symmetric function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$. We define *the re-ordered functional derivative of the m-th order with respect to $C = C(x, y)$*

$$R_C\left[\frac{\delta^m}{\delta f(x_1)\cdots\delta f(x_m)}\right] Z(f)$$

by the following recursion relation:

$$R_C\left[\frac{\delta}{\delta f(x)}\right] Z(f) = \frac{\delta Z(f)}{\delta f(x)}, \tag{2.5}$$

$$\begin{aligned} &R_C\left[\frac{\delta^m}{\delta f(x_1)\cdots\delta f(x_m)}\right] Z(f) \\ &\quad = \frac{\delta}{\delta f(x_1)} R_C\left[\frac{\delta^{m-1}}{\delta f(x_2)\cdots\delta f(x_m)}\right] Z(f) \\ &\qquad + \sum_{k=2}^{m} C(x_1, x_k) R_C\left[\frac{\delta^{m-2}}{\delta f(x_2)\cdots\delta f(x_{k-1})\delta f(x_{k+1})\cdots\delta f(x_m)}\right] Z(f), \\ &\qquad\qquad m = 2, 3, \cdots. \end{aligned} \tag{2.6}$$

Let $K(\boldsymbol{R}^n)$ be the set of functions $\rho \in C_0^\infty(\boldsymbol{R}^n)$, satisfying

$$0 \leqslant \rho, \quad 0 < \rho(0), \quad \int dx \rho(x) = 1.$$

For $\rho \in K(\boldsymbol{R}^n)$ and $\kappa > 0$ we define

$$\rho_\kappa(x) = \kappa^n \rho(\kappa x), \tag{2.7}$$

which tends to n-dimensional δ-function in the distribution sense as $\kappa \to \infty$.

Definition 2.4. Let $Z(f)$ and $C(x, y)$ be as in Definition 2.3. We define the distribution on $\boldsymbol{R}^n$

$$R_C\left[\frac{\delta^m}{\delta f(x)^m}\right]Z(f)$$

by

$$R_C\left[\frac{\delta^m}{\delta f(x)^m}\right]Z(f)=\lim_{\kappa\to\infty}\int dx_1\cdots dx_m R_C\left[\frac{\delta^m}{\delta f(x_1)\cdots\delta f(x_m)}\right]Z(f) \times \rho_\kappa(x-x_1)\cdots\rho_\kappa(x-x_m), \tag{2.8}$$

provided that the right hand side exists in the distribution sense independently of any choices of $\rho \in K(\boldsymbol{R}^n)$.

3. The equation and the result. The Schwinger equation we consider is:

$$(-\Delta+M^2)\frac{\delta Z(f)}{\delta f(x)}=-f(x)Z(f)+\sum_{k=1}^{p}(-1)^k a_k R_C\left[\frac{\delta^{2k-1}}{\delta f(x)^{2k-1}}\right]Z(f) \tag{3.1}$$

with the subsidary condition

$$Z(0)=1. \tag{3.2}$$

Here Δ is the n-dimensional Laplacian and $p>1$ is an arbitrary fixed number with $a_p>0$, $a_k \in \boldsymbol{R}$, $k=1, 2, \cdots, p-1$. $M>0$ is a parameter. We take $C(x, y)$ equal to the integral kernel of the bounded self-adjoint operator

$$C=(-\Delta+M^2)^{-1} \tag{3.3}$$

on $L^2(\boldsymbol{R}^n)$. We seek a solution to Eq. (3.1) in the space of functionals on $\mathcal{S}_r(\boldsymbol{R}^n)$, the space generated by real functions in $\mathcal{S}(\boldsymbol{R}^n)$. The functional derivatives for functionals on $\mathcal{S}_r(\boldsymbol{R}^n)$ are defined in the same way as those for functionals on $\mathcal{S}(\boldsymbol{R}^n)$.

Our result is:

Theorem. *Let $n=1$ or 2. Then, there exists a solution $Z=Z(f)$ to Eq.* (3.1) *with* (3.2), *which is written as the characteristic functional of a probability measure μ on $\mathcal{S}'_r(\boldsymbol{R}^n)$*:

$$Z(f)=\int_{\mathcal{S}'_r(\boldsymbol{R}^n)} e^{i(T,f)}d\mu(T), \qquad f\in \mathcal{S}_r(\boldsymbol{R}^n). \tag{3.4}$$

Further, Z has functional derivatives of all orders:

$$\frac{\delta^m Z(f)}{\delta f(x_1)\cdots\delta f(x_m)}=i^m\int_{\mathcal{S}'_r(\boldsymbol{R}^n)} T(x_1)\cdots T(x_m)e^{i(T,f)}d\mu(T), \qquad m=1, 2, \cdots. \tag{3.5}$$

The proof, which is based on the methods and results of the so-called constructive quantum field theory (see, e.g., Glimm and Jaffe [3] and Simon [6]) and rather lengthy, will be given elsewhere.

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