61. A Result on the Siegel-Ramachandra Class Invariant over Imaginary Quadratic Fields

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Let K be an imaginary quadratic field embedded in the complex number field $\boldsymbol{C}$, and let $\mathfrak{f}\neq(1)$ be an integral ideal of K. For each element C of the ray class group $Cl(\mathfrak{f})$ modulo $\mathfrak{f}$ the ray class invariant $\phi_{\mathfrak{f}}(C)$ is introduced by Siegel and Ramachandra (cf. Robert [5] and Stark [7]). The definition will be explained in the text. Let $H(\mathfrak{f})$ be the ray class field modulo $\mathfrak{f}$ and f the smallest positive integer contained in $\mathfrak{f}$. Then it is known that $(\phi_{\mathfrak{f}}(C)/\phi_{\mathfrak{f}}(C'))^{e(H(\mathfrak{f}))/e(K)}$ $(C, C'\in Cl(\mathfrak{f}))$ is the $(12f)$-th power of a unit of $H(\mathfrak{f})$ (Gillard and Robert [1]). Here $e(H(\mathfrak{f}))$ and $e(K)$ are the numbers of roots of unity contained in $H(\mathfrak{f})$ and K respectively. In this paper we describe a $(12f)$-th root of $(\phi_{\mathfrak{f}}(C)/\phi_{\mathfrak{f}}(C'))^{e(H(\mathfrak{f}))/e(K)}$ contained in $H(\mathfrak{f})$ explicitly by special values of the Siegel functions and determine the behavior under Artin automorphisms. The result is then useful to calculate class numbers of abelian extensions of K by the method of Gras and Gras [2].

§1. **Preliminaries.** Let $\mathfrak{f}\neq(1)$ be an integral ideal of K. The ideal $\mathfrak{f}$ is uniquely decomposed into two factors $\mathfrak{f}_a$, $\mathfrak{f}_b$ as follows:

$$\mathfrak{f}=\mathfrak{f}_a\mathfrak{f}_b,\quad \mathfrak{f}_a=\bar{\mathfrak{f}}_a,\quad (\mathfrak{f}_b,\bar{\mathfrak{f}}_b)=1.$$

Here the bar indicates the complex conjugation. Take an integral basis $\{\omega, 1\}$ $(\mathrm{Im}(\omega)>0)$ of the ring $\mathfrak{o}$ of integers of K. We fix such an ω throughout this paper. The next lemma is fundamental in the formulation and the proof of our results.

Lemma. *Let f_b be the smallest positive integer contained in $\mathfrak{f}_b$. Then there exists a rational integer a satisfying the following condition: For an arbitrary element x of $\mathfrak{f}_b$ the congruence*

$$a\,\mathrm{tr}(x)\equiv \mathrm{Im}(x)/\mathrm{Im}(\omega)\mod f_b$$

holds, where $\mathrm{tr}(\cdot)$ *is the trace map from K to the rational number field $\boldsymbol{Q}$.*

We fix such an integer a. For an algebraic number field H of finite degree denote by $e(H)$ the number of roots of unity contained in H. Put $\delta=e(H(1))/e(K)$, where $H(1)$ is the Hilbert class field of K. The integer δ is a divisor of 6. We consider the following condition (♯) concerning an ideal (not necessarily integral) $\mathfrak{a}$ of K:

(♯) $\mathfrak{a}$ is prime to $6\mathfrak{f}$ and $N(\mathfrak{a})\equiv 1 \mod (12/\delta)$.

Here $N(\mathfrak{a})$ is the absolute norm of $\mathfrak{a}$ and the congruence is considered multiplicatively. Every absolute ideal class of K contains infinitely many prime ideals of degree one satisfying the condition (♯). Let t and z be complex numbers with $\mathrm{Im}(z)>0$, and put $e(t)=\exp(2\pi it)$. Define the Siegel function $g(t,z)$ by

$$g(t,z)=2e\left(\frac{z}{12}+\frac{t\,\mathrm{Im}(t)}{2\,\mathrm{Im}(z)}\right)\sin(\pi t)\prod_{m=1}^{\infty}(1-2\cos(2\pi t)e(mz)+e(2mz)).$$

Cf. [5] and [7].

For an ideal $\mathfrak{a}$ of K satisfying (♯), take an integral basis $\{\mu,\nu\}$ ($\mathrm{Im}(\mu/\nu)>0$) of $\mathfrak{a}$ and put

$$s_{\mathfrak{f}}(t,\mathfrak{a})=[e(a|t|^2/N(\mathfrak{a}))\lambda(A)g(t/\nu,\mu/\nu)]^{e(H(\mathfrak{f}))/e(K)}.$$

Here $\lambda(A)$ is a 12-th root of unity defined as follows. Take another integral ideal $\mathfrak{c}$ with (♯) contained in $\mathfrak{a}$, and let $\{\rho,\tau\}$ ($\mathrm{Im}(\rho/\tau)>0$) be an integral basis of $\mathfrak{c}$. Define 2×2 matrices B_1 and B_2 by

$$\begin{pmatrix}\rho\\ \tau\end{pmatrix}=B_1\begin{pmatrix}\omega\\ 1\end{pmatrix}=B_2\begin{pmatrix}\mu\\ \nu\end{pmatrix},$$

and choose integral matrices A_1 and A_2 so that

$$\det(A_i)=1,\quad A_i\equiv B_i \bmod (12/\delta)\quad (i=1,2).$$

Put $A=A_1^{-1}A_2=\begin{pmatrix}* & *\\ c & d\end{pmatrix}$ and let $\lambda(A)$ be the 12-th root of unity appearing in the transformation formula of the Dedekind eta-function:

$$\eta^2(Az)=\lambda(A)(cz+d)\eta^2(z).$$

By the transformation property of $g(t,z)$ under the modular group (cf. [7]) and by the fact that δ divides $e(H(\mathfrak{f}))/e(K)$, we see that the above definition is well-defined. The symbol $s_{\mathfrak{f}}(t,\mathfrak{a})$ depends on the choice of the integer a and the integral basis $\{\omega,1\}$, but we do not indicate them for simplicity.

§ 2. **Main results.** Denote by $A(\mathfrak{f})$ the set of pairs $(t,\mathfrak{a})$ of $t\in K^{\times}$ and an ideal $\mathfrak{a}$ of K satisfying (♯) such that $t^{-1}\mathfrak{a}\cap\mathfrak{o}=\mathfrak{f}$. Every element $(t,\mathfrak{a})$ of $A(\mathfrak{f})$ gives an integral ideal $t\mathfrak{a}^{-1}\mathfrak{f}$ prime to $\mathfrak{f}$, whose class in $Cl(\mathfrak{f})$ is denoted by $C(t,\mathfrak{a})$. By the fact mentioned after the condition (♯), the map $A(\mathfrak{f})\ni(t,\mathfrak{a})\mapsto C(t,\mathfrak{a})\in Cl(\mathfrak{f})$ is surjective. Take an element γ of $K^{\times}$ and an integral ideal $\mathfrak{g}$ satisfying (♯) so that $\mathfrak{f}=\gamma\mathfrak{g}^{-1}$. Every element C of $Cl(\mathfrak{f})$ is represented by an integral ideal of the form $\alpha\mathfrak{b}^{-1}$, where $\mathfrak{b}$ is an integral ideal satisfying (♯) and $\alpha\in\mathfrak{b}$, and is written as $C=C(\alpha/\gamma,\mathfrak{b}/\mathfrak{g})$. Let $\{\mu,\nu\}$ ($\mathrm{Im}(\mu/\nu)>0$) be a basis of $\mathfrak{b}/\mathfrak{g}$. Then the Siegel-Ramachandra class invariant $\phi_{\mathfrak{f}}(C)$ is defined by

$$\phi_{\mathfrak{f}}(C)=g(\alpha/\gamma\nu,\mu/\nu)^{12f}.$$

Theorem. *Let $\mathfrak{f}\neq(1)$ be an integral ideal of K prime to 6, and express $\mathfrak{f}=\gamma\mathfrak{g}^{-1}$ as above. Let $(\alpha/\gamma,\mathfrak{b}/\mathfrak{g})$ and $(\alpha'/\gamma,\mathfrak{b}'/\mathfrak{g})$ be two elements of $A(\mathfrak{f})$ with integral ideals $\mathfrak{b}$ and $\mathfrak{b}'$. Assume $\alpha\equiv\alpha' \bmod 2$. Then we have:*

(0) $s_{\mathfrak{f}}(\alpha/\gamma,\mathfrak{b}/\mathfrak{g})^{12f}=\phi_{\mathfrak{f}}(C)^{e(H(\mathfrak{f}))/e(K)}$, *where* $C=C(\alpha/\gamma,\mathfrak{b}/\mathfrak{g})$.

(i) $s_{\mathfrak{f}}(\alpha/\gamma, \mathfrak{b}/\mathfrak{g})$ *is an algebraic integer of the ray class field modulo* $12\mathfrak{f}$.

(ii) $s_{\mathfrak{f}}(\alpha/\gamma, \mathfrak{b}/\mathfrak{g})/s_{\mathfrak{f}}(\alpha'/\gamma, \mathfrak{b}'/\mathfrak{g})$ *is a unit of* $H(\mathfrak{f})$.

(iii) *For an arbitrary ideal* $\mathfrak{b}_0$ *satisfying* ($\sharp$) *and* $\alpha_0 \in \mathfrak{b}_0$, *we have*

$$[s_{\mathfrak{f}}(\alpha/\gamma, \mathfrak{b}/\mathfrak{g})/s_{\mathfrak{f}}(\alpha'/\gamma, \mathfrak{b}'/\mathfrak{g})]^{(\alpha_0\mathfrak{b}_0^{-1}, H(\mathfrak{f})/K)} = s_{\mathfrak{f}}(\alpha\alpha_0/\gamma, \mathfrak{b}\mathfrak{b}_0/\mathfrak{g})/s_{\mathfrak{f}}(\alpha'\alpha_0/\gamma, \mathfrak{b}'\mathfrak{b}_0/\mathfrak{g})$$

if $\alpha_0\mathfrak{b}_0^{-1}$ *is prime to* $\mathfrak{f}$. *Here* $(\cdot, H(\mathfrak{f})/K)$ *is the Artin symbol for* $H(\mathfrak{f})$.

Remark. When $\mathfrak{b}=\mathfrak{b}'$, the assertions (ii) and (iii) are valid whithout the assumption $(\mathfrak{f}, 6)=1$.

§ 3. Index-class number formula. Suppose $(\mathfrak{f}, 6)=1$ and let $C_1, \cdots, C_h$ be elements of $Cl(\mathfrak{f})$. Write $C_i = C(\alpha_i/\gamma, \mathfrak{b}_i/\mathfrak{g})$ $(i=1, \cdots, h)$ as explained in § 2. We assume that all α_i's $(i=1, \cdots, h)$ are congruent to each other modulo 2. Then by the theorem, we see that

$$s_{\mathfrak{f}}(\alpha_i/\gamma, \mathfrak{b}_i/\mathfrak{g})/s_{\mathfrak{f}}(\alpha_1/\gamma, \mathfrak{b}_1/\mathfrak{g}) \qquad (i=2, \cdots, h)$$

generate a subgroup S of the unit group of $H(\mathfrak{f})$. Let H be an abelian extension of K whose conductor is $\mathfrak{f}$. We define a subgroup $S(H)$ of the unit group $E(H)$ of H by

$$S(H)=\mu(H)\times N_{H(\mathfrak{f})/H}(S),$$

where $\mu(H)$ is the torsion part of $E(H)$ and $N_{H(\mathfrak{f})/H}(\cdot)$ is the norm map from $H(\mathfrak{f})$ to H. By the analytic class number formula, we obtain the following proposition.

Proposition. (i) *Let* $\mathfrak{p}$ *be a prime ideal of* K *such that* $\bar{\mathfrak{p}}\neq\mathfrak{p}$ *and* $\mathfrak{p}\nmid 6$. *Suppose that* $\mathfrak{f}=\mathfrak{p}^n$ $(n>0)$ *and* $H\cap H(1)=K$. *Then we have*

$$(E(H):S(H))=\delta^{[H:K]-1}h_H/h_K,$$

where h_H *and* h_K *are the class numbers of* H *and* K *respectively, and* $\delta=e(H(1))/e(K)$.

(ii) *If* $[H:K]$ *is a prime number* p,

$$(E(H):S(H))=(f_a\delta)^{p-1}(e(K)/e(H))h_H/h_K,$$

where f_a *is the smallest positive integer in* $\mathfrak{f}_a$.

Remark. We can obtain more general formulas following Nakamula [4] or Schertz [6].

Example. Let $K=\boldsymbol{Q}(\sqrt{-19})$, $\mathfrak{p}=(\gamma)$ $(\gamma=\omega+1,\ \omega=(1+\sqrt{-19})/2)$, and let H be the ray class field modulo $\mathfrak{p}$. The ideal $\mathfrak{p}$ is a prime ideal over 7. The field H is a cubic cyclic extension of K, the Galois group G of which is generated by $\sigma=((3), H/K)$. Note that H is not Galois over $\boldsymbol{Q}$ and that our result is conveniently applied in this type of situation. In the present case, we have

$$(E(H):S(H))=h_H, \qquad S(H)=\{\pm 1\}\times\varepsilon^{Z[G]}.$$

Here the unit ε and its conjugates over K are as follows:

$$\varepsilon=s_{\mathfrak{p}}(3/\gamma, \mathfrak{o})/s_{\mathfrak{p}}(1/\gamma, \mathfrak{o}),$$
$$\varepsilon^{\sigma}=s_{\mathfrak{p}}(9/\gamma, \mathfrak{o})/s_{\mathfrak{p}}(3/\gamma, \mathfrak{o}),$$
$$\varepsilon^{\sigma^2}=s_{\mathfrak{p}}(27/\gamma, \mathfrak{o})/s_{\mathfrak{p}}(9/\gamma, \mathfrak{o}).$$

If we take the integer a in the definition of s_p to be -2, the minimal polynomial of ε over K is $X^3-2X^2+\omega X+1$. To determine this, we used the approximations

$$\varepsilon \sim 2.11673140 - i1.06052873,$$
$$\varepsilon^{\sigma} \sim 0.30333461 + i0.70624358,$$
$$\varepsilon^{\sigma^2} \sim -0.42006601 + i0.35428514.$$

After some considerations following Gras and Gras [2] (see also [4]), we conclude that $E(H)=S(H)$ and $h_H=1$ in this case.

References

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