59. On an Anderson-Anderson Problem

By Ryûki MATSUDA
Department of Mathematics, Faculty of Science, Ibaraki University

(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1983)

§1. **Introduction.** Let R be a commutative integral domain with quotient field K. For nonzero fractional ideals I and J, we define $I:J=\{x \in R; xJ \subset I\}$. We will denote $\{x \in K; xI \subset R\}$ by I^{-1}, and $(I^{-1})^{-1}$ by I_v. We will say that I is a *divisorial ideal* or *v-ideal* if $I=I_v$. I is a v-ideal of *finite type* if $I=J_v$ for some finitely generated fractional ideal J. By a *graded domain* $R=\oplus_{\alpha\in\Gamma} R_\alpha$, we mean an integral domain R graded by an arbitrary torsionless grading monoid Γ, i.e., a commutative cancellative monoid, written additively, such that the quotient group $\langle\Gamma\rangle$ generated by Γ is a torsion-free abelian group. (A general reference on torsionless grading monoids and Γ-graded rings is [5].) For a fractional ideal I of a Γ-graded integral domain $R=\oplus_{\alpha\in\Gamma} R_\alpha$, I^* will denote the fractional ideal generated by the homogeneous elements of I. Let $x \in R$, with $x=x_1+x_2+\cdots+x_n$, where $x_i \in R_{\alpha_i}$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. We then define the content of x, denoted by $C(x)$, to be $(x_1, x_2, \cdots, x_n)$. One of the most important examples of a Γ-graded integral domain is the semigroup ring $R[X;\Gamma]$. Here $R[X;\Gamma]=R[\{X^g; g\in\Gamma\}]$ with $X^gX^h=X^{g+h}$. $R[X;\Gamma]$ is Γ-graded in the natural way with $\deg(X^g)=g$. In [1], D. D. Anderson-D. F. Anderson studied v-ideals and invertible ideals of a Γ-graded domain R. Specifically in §3 they gave necessary and sufficient conditions for an integral v-ideal of R to be homogeneous whenever it contains a nonzero homogeneous element, proving the following result.

Theorem ([1], Theorem 3.2). *Let $R=\oplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and suppose $S=\{$nonzero homogeneous elements of $R\}\neq\phi$. The following statements are equivalent.*

(1) *For $r \in S$ and $x \in R$, $(r):(x)$ is homogeneous.*

(2) *If I is an integral v-ideal of R with nonzero I^*, then I is homogeneous.*

(3) *If I is an integral v-ideal of R of finite type with nonzero I^*, then I is homogeneous.*

(4) *$C(xy)_v=(C(x)C(y))_v$ for all nonzero $x, y \in R$.*

(5) *For each nonzero $x \in R$, $xR_S \cap R = xC(x)^{-1}$.*

(6) *If I is an integral v-ideal of R of finite type, then $I=qJ$ for some $q \in R_S$ and some homogeneous integral v-ideal J of R of finite type.*

They asked in [1] if in (6) it is necessary to assume that I is of finite type. In this paper we show that this is actually necessary for an infinite number of graded integral domains which we should construct.

Let A be a commutative ring, and let Γ be a commutative cancellative monoid. In Appendix we give a necessary and sufficient condition for semigroup ring $A[X;\Gamma]$ to be a Noetherian ring.

§2. **Examples.** Let D be a domain of characteristic $p>0$, and let Γ be an additive subgroup of the real numbers $\boldsymbol{R}$ containing $\{1/p, 1/(p^2), 1/(p^3), \cdots\}$. We denote the group ring $D[X;\Gamma]$ by R throughout the section.

Lemma 1. *Let $\{d(1), d(2), d(3), \cdots\}$ and $\{n(1), n(2), n(3), \cdots\}$ be two sequences of natural numbers (≥ 1). Set $f_n=\prod_{i=1}^{n}(1-X^{n(i)/p^{d(i)}})$ for each natural number n. Suppose that (1) $d(1)<d(2)<d(3)<\cdots$, and (2) $n(i)<p$ for each natural number i. Then the ideal $\bigcap_{i=1}^{\infty}(f_i)$ of R is not zero.*

Proof. Let k and l be natural numbers such that $k<l$. We set $e=(p-1)!$ and set

$$e(k,l)=p^{d(l)-d(l)}+p^{d(l)-d(l-1)}+\cdots+p^{d(l)-d(k)}.$$

Then we have

$$(*)\qquad e(k,l)\leq(p^{d(l)-d(k)+1}-1)/(p-1).$$

Since $1-X^{e/p^{d(i)}}=(1-X^{e/p^{d(l)}})^{p^{d(l)-d(i)}}$, we have

$$\prod_{i=k}^{l}(1-X^{e/p^{d(i)}})=(1-X^{e/p^{d(l)}})^{e(k,l)}.$$

Since $1-X^{n(i)/p^{d(i)}}$ divides $1-X^{e/p^{d(i)}}$ in R, $\prod_{i=k}^{l}(1-X^{n(i)/p^{d(i)}})$ divides $(1-X^{e/p^{d(l)}})^{e(k,l)}$. By $(*)$ we see that $(1-X^{e/p^{d(l)}})^{e(k,l)}$ divides $1-X^{e/p^{d(k)-1}}$. Therefore f_l divides $1-X^{e/p^{d(1)-1}}$. Hence $1-X^{e/p^{d(1)-1}}\in\bigcap_{i=1}^{\infty}(f_i)$.

Let $f=a_1X^{\alpha_1}+a_2X^{\alpha_2}+\cdots+a_nX^{\alpha_n}$ be a nonzero element of R, where $0\neq a_i\in D$ for each i and $\alpha_1<\alpha_2<\cdots<\alpha_n$. We set $\alpha_1=\operatorname{ord}(f)$ (resp. $\alpha_n=\deg(f)$), and call it the *order* (resp. *degree*) of f.

Lemma 2. *In Lemma 1, suppose that $\bigcap_{i=1}^{\infty}(f_i)$ is a principal ideal (f) of R. Then we have $f\neq 0$ and*

$$\sum_{i=1}^{\infty}n(i)/p^{d(i)}=\deg(f)-\operatorname{ord}(f).$$

Proof. By Lemma 1 we have $f\neq 0$. We may assume that the order of f is zero. We set $\deg(f)=d$. Since $f\in(f_l)$ for each l, we have

$$n(1)/p^{d(1)}+n(2)/p^{d(2)}+\cdots+n(l)/p^{d(l)}\leq d.$$

It follows that $\sum_{i=1}^{\infty}n(i)/p^{d(i)}\leq d$. Since $\prod_{i=k}^{l}(1-X^{n(i)/p^{d(i)}})$ divides $(1-X^{e/p^{d(l)}})^{e(k,l)}$ in the proof of Lemma 1, we see that f_l divides

$$(*)\quad(1-X^{n(1)/p^{d(1)}})(1-X^{n(2)/p^{d(2)}})\cdots(1-X^{n(k-1)/p^{d(k-1)}})(1-X^{e/p^{d(l)}})^{e(k,l)}.$$

Hence f divides $(*)$. It follows that

$$d\leq n(1)/p^{d(1)}+n(2)/p^{d(2)}+\cdots+n(k-1)/p^{d(k-1)}+ee(k,l)/p^{d(l)}.$$

By (*) of the proof of Lemma 1, we have

$$d \leq n(1)/p^{d(1)} + n(2)/p^{d(2)} + \cdots + n(k-1)/p^{d(k-1)} + ep^{1-d(k)}/(p-1).$$

It follows that

$$d \leq \sum_{i=1}^{\infty} n(i)/p^{d(i)}.$$

We see that $\sum_{i=1}^{\infty} n(i)/p^{d(i)} = d$.

Lemma 3. *Suppose that D is a field. Then R satisfies six conditions of Theorem of* § 1.

Proof. Let $r \in S$, and let $x \in R$. r is a unit element of R. It follows that $(r):(x)=R$. Therefore R satisfies the condition (1) of Theorem.

Lemma 4. *Suppose that $G \subsetneq \boldsymbol{R}$. Then there exist two sequences* $\{d(1), d(2), d(3), \cdots\}$ *and* $\{n(1), n(2), n(3), \cdots\}$ *of natural numbers such that* (1) $\{d(1), d(2), d(3), \cdots\}$ (resp. $\{n(1), n(2), n(3), \cdots\}$) *satisfies the condition* (1) (resp. (2)) *of Lemma* 1, *and* (2) $\sum_{i=1}^{\infty} n(i)/p^{d(i)} \in \boldsymbol{R}-G$.

Proof. Since $1/p \in G$, we have $G \supset Z$. We choose $\alpha \in \boldsymbol{R}-G$. Since $G \supset Z$, we may take $0<\alpha<1$. The p-adic expression of α gives desired sequences of natural numbers.

Lemma 5. *In Lemma 4, we set $f_n = \prod_{i=1}^{n}(1-X^{n(i)/p^{d(i)}})$ for each n. Then the ideal $\bigcap_{i=1}^{\infty}(f_i)$ of R is not principal.*

Proof. Suppose that $\bigcap_{i=1}^{\infty}(f_i)$ is a principal ideal (f). By Lemma 2 we have $f \neq 0$ and $\sum_{i=1}^{\infty} n(i)/p^{d(i)} = \deg(f) - \operatorname{ord}(f)$. Since $\deg(f) \in G$ and $\operatorname{ord}(f) \in G$, it follows $\sum_{i=1}^{\infty} n(i)/p^{d(i)} \in G$. This contradicts to the condition (2) of Lemma 4.

Lemma 6 ([4], Corollary 3.1). *Suppose that D is a field and that G is contained in the additive group of rational numbers. Then R is a Bezout domain* (*that is, each finitely generated ideal of R is a principal ideal*).

§ 3. **Main theorem.** Now we can answer to Anderson-Anderson problem by the following theorem.

Theorem. *There exists an infinite number of graded integral domains R such that* (1) *R is a group ring,* (2) *R satisfies six conditions of Theorem of* § 1, (3) *There exists an integral v-ideal I of R which is never of the form qJ, where $q \in R_S$ and J is a homogeneous integral v-ideal of R of finite type.*

Proof. We take a group ring R of Lemma 6. By Lemma 3, R satisfies the condition (2). We take a pair of sequences

$$\{d(1), d(2), d(3), \cdots\} \quad \text{and} \quad \{n(1), n(2), n(3), \cdots\}$$

of Lemma 4. We set $f_n = \prod_{i=1}^{n}(1-X^{n(i)/p^{d(i)}})$ for each n and set $I = \bigcap_{i=1}^{\infty}(f_i)$. Then I is an integral v-ideal of R([2], § 1, n°1). Suppose that I is of the form qJ for some $q \in R_S$ and some homogeneous integral v-ideal J of R of finite type. Since S consists of unit elements of R, I is a finitely generated ideal of R. By Lemma 6, we see that I is a principal ideal of R. This contradicts to Lemma 5.

§ 4. **Appendix.** Let A be a commutative ring with identity, and let Γ be a commutative cancellative monoid. (We note that Γ is not necessarily torsion-free.) If Γ is a group, I. Connell ([3], Theorem 2, (c)) proved that $A[X;\Gamma]$ is a Noetherian ring if and only if A is a Noetherian ring and Γ is a finitely generated group. We prove now the following result which applies also to the case where Γ is not a group.

Theorem. *$A[X;\Gamma]$ is a Noetherian ring, if and only if A is a Noetherian ring and Γ is a finitely generated monoid.*

Proof. Assume that $A[X;\Gamma]$ is a Noetherian ring. Then A is clearly a Noetherian ring. A chain of ideals in Γ gives a chain of ideals in $A[X;\Gamma]$. Hence Γ has the ascending chain condition on ideals. Since Γ is also cancellative, every element is a sum of irreducible elements. If $c_1, c_2, c_3, \cdots$ are irreducible elements of Γ, then $Z_0c_1 \subset Z_0c_1+Z_0c_2 \subset Z_0c_1+Z_0c_2+Z_0c_3 \subset \cdots$ is a chain of ideals in Γ where Z_0 is the nonnegative integers; hence all c_i are in some $Z_0c_1+Z_0c_2+\cdots+Z_0c_n$. As Γ is cancellative, each c_i is a unit times some one of $c_1, c_2, \cdots, c_n$. Thus there are only finitely many irreducible elements up to units. Let H be the group of units of Γ. If J is an ideal of $A[X;H]$, then $J+A[X;\Gamma-H]$ is an ideal in $A[X;\Gamma]$. Hence $A[X;H]$ is a Noetherian ring. By the result of Connell, H is a finitely generated group. Therefore Γ is a finitely generated monoid. The sufficiency is clear.

References

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