58. On Representations of $\mathfrak{p}$-Adic Split and Non-Split Symplectic Groups and their Character Relations

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0. **Introduction.** Let F be a non-archimedean local field, $\bar{F}$ its algebraic closure, and D a division quaternion algebra over F. Then, up to local isomorphisms over F, there are two algebraic groups over F which are $\bar{F}$-isomorphic to $GSp(n)$, a symplectic group of genus n with similitudes. They are $GSp(n)$ and $GUq(n)$, the latter being the quaternionic unitary group of size n with similitudes.

Jacquet and Langlands stated in [3] that there exists a 'good' correspondence in terms of characters between the irreducible admissible representations of $GSp(1, F)\cong GL(2, F)$ and those of $GUq(1, F)\cong D^{\times}$. Our main purpose is to find a good correspondence between the admissible representations of $GSp(2, F)$ and those of $GUq(2, F)$. This is a representation-theoretic approach to a problem raised in Y. Ihara [2]: Are there any connections between Dirichlet series attached to spherical functions of $USp(2)$ and those attached to Siegel modular forms of degree two?

We set

$$G=GSp(2, F)=\{g \in GL(4, F)\,;\, gJ{}^tg=n(g)J,\ n(g)\in F^{\times}\},$$

$$G^*=GUq(2, F)=\left\{g \in GL(2, D)\,;\, g\begin{pmatrix} & 1\\ 1 & \end{pmatrix}{}^t\bar{g}=n(g)\begin{pmatrix} & 1\\ 1 & \end{pmatrix} n(g)\in F^{\times}\right\},$$

where $J=\begin{pmatrix} & I_2\\ -I_2 & \end{pmatrix}$ and '$-$' denotes the main involution of D.

In this note, we first classify the conjugacy classes of maximal F-tori (i.e. tori defined over F) of G and those of G^*. Then we define some induced representations of G and G^*, and calculate their distributive characters. Finally, we state our main result which says that there are 'good' character relations between some types of induced representations of G and those of G^*. Details will be published elsewhere.

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In what follows, we assume that F has odd residue characteristic.

1. There are five types of F-conjugacy classes of maximal F-tori of G. Let T be any maximal F-torus of G. Then $T^{(1)}(F)$, the group of F-rational points of T with similitudes one, is F-isomorphic to one of the following five types of groups:

(1) $F^{\times}\times F^{\times}$

(2) $F^{\times}\times\{\alpha\in E\,;\ \mathrm{Nr}_{E/F}(\alpha)=1\}$

(3) $E^{\times}$

(4) $\{(\alpha,\beta)\in E_1^{\times}\times E_2^{\times}\,;\ \mathrm{Nr}_{E_1/F}(\alpha)=\mathrm{Nr}_{E_2/F}(\beta)=1\}$

(5) $\{\alpha\in K\,;\ \mathrm{Nr}_{K/E}(\alpha)=1\}$,

where E_1, E_2 and E are separable quadratic extensions of F, and $K\supset E\supset F$ is a tower of separable quadratic extensions. Note that the groups (1), (2), (3) are non-compact and (4), (5) are compact.

As for G^*, there are three types of F-conjugacy classes of maximal F-tori. Let T^* be any maximal F-torus of G^*. Then $T^{*(1)}(F)$, the group of F-rational points of T^* with similitudes one, is F-isomorphic to one of the three types of groups (3), (4), (5) above. If $T^{(1)}(F)$ and $T^{*(1)}(F)$ are F-isomorphic, we say that T and T^* are *corresponding* F-tori of G and G^*. The irreducible admissible representations of G and G^* which are 'parametrized' (in the sense of Harish-Chandra) by the duals of the two corresponding F-tori of types (4) and (5) are related to absolutely cuspidal representations. We shall exclusively consider the representations of G and G^* which are parametrized by the duals of the corresponding *F-tori of type* (3). The torus of G belonging to type (1) (i.e. maximal F-split torus) will be called T_1, and the tori of G and G^* belonging to type (3) will be called $T_{3,E}$, $T^*_{3,E}$ respectively, where E is the corresponding quadratic field over F. Namely

$$T_1(F)=\left\{t_1=\begin{pmatrix} a & & & \\ & b & & \\ & & sa^{-1} & \\ & & & sb^{-1}\end{pmatrix};\ a,\ b,\ s\in F^{\times}\right\}$$

$$T_{3,E}(F)=\left\{t_3=\begin{pmatrix} A & \\ & s\,{}^tA^{-1}\end{pmatrix};\ s\in F^{\times},\ A\in i_E(E^{\times})\subset GL(2,F)\right\}$$

$$T^*_{3,E}(F)=\left\{t_3^*=\begin{pmatrix} \alpha & \\ & s\bar{\alpha}^{-1}\end{pmatrix};\ s\in F^{\times},\ \alpha\in i_E^*(E^{\times})\subset D^{\times}\right\},$$

where i_E and i_E^* are fixed F-isomorphisms of $E^{\times}$ into $GL(2,F)$ and $D^{\times}$, respectively. We denote by W_1, $W_{3,E}$ and $W^*_{3,E}$ the Weyl groups of T_1, $T_{3,E}$, and $T^*_{3,E}$, respectively.

2. We define the following parabolic subgroups of G and G^* in order to introduce some induced representations:

$$P=\left\{p=\begin{pmatrix} A & * \\ 0 & s\,{}^tA^{-1}\end{pmatrix};\ A\in GL(2,F),\ s\in F^{\times}\right\},$$

$$P^*=\left\{p^*=\begin{pmatrix}\alpha & * \\ 0 & s\bar{\alpha}^{-1}\end{pmatrix};\ \alpha\in D^\times,\ s\in F^\times\right\}.$$

The Levi subgroups of P and P^* contain maximal F-tori of type (3). Let W be a vector space over C. Then we denote by $F(G, W)$ the set of all locally constant W-valued functions on G. Let $R(GL(2, F))$ be the set of all equivalence classes of irreducible admissible representations of $GL(2, F)$. For each $(\pi, V)\in R(GL(2, F))$, $(r, U)\in R(D^\times)$ and $\eta\in R(F^\times)$, we define the following representation spaces of G and G^*:

$$B(\pi, \eta)=\{f\in F(G, V);\ f(pg)=\eta(s)\delta^{-1/2}(p)\pi(A)f(g),\ p\in P,\ g\in G\}$$

$$B^*(r, \eta)=\{f\in F(G^*, U);\ f(p^*g)=\eta(s)\delta^{*-1/2}(p^*)r(\alpha)f(g),\ p^*\in P^*,\ g\in G^*\},$$

where p and p^* are expressed as in the above form, and

$$\delta(p)=|s\cdot\det(A)^{-1}|^3,\qquad \delta^*(p^*)=|s\cdot(\alpha\bar{\alpha})^{-1}|^3.$$

Here $|\cdot|$ denotes the normalized absolute value of F.

The groups G and G^* act on vector spaces $B(\pi, \eta)$ and $B^*(r, \eta)$ by right translations, respectively. These induced representations are admissible. We denote them by $\rho(\pi, \eta)$ and $\rho^*(r, \eta)$. These representations will be irreducible if (π, V) and (r, U) are sufficiently 'general'.

Let $\mathcal{S}(G)$ be the set of all C-valued functions on G which are locally constant and compactly supported. For $f\in\mathcal{S}(G)$ and $(\Pi, W)\in R(G)$, put

$$\Pi(f)=\int_G f(g)\Pi(g)dg\in\operatorname{End}(W),$$

where dg is a Haar measure on G. Since Π is admissible, the operator $\Pi(f)$ has a finite range. Hence the trace $\Pi(f)$ is defined. If there exists a locally integrable function $\mathcal{X}_\Pi$ on G such that

$$\operatorname{trace}\Pi(f)=\int_G f(g)\mathcal{X}_\Pi(g)dg$$

holds for any $f\in\mathcal{S}(G)$, then we call this function the character of Π. Let $X^2-\sigma X+\tau\in F[X]$ be the characteristic polynomial of $A\in GL(2, F)$, then we put $d(A)=|(\sigma^2-4\tau)\tau^{-1}|$. Let $X^4-AX^3+BX^2-sAX+s^2\in F[X]$ be the characteristic polynomial of $g\in G$ where $n(g)=s$, then we put $D(g)=|\{A^2-4(B-2s)^2\}\cdot\{(B+2s)^2-4sA^2\}\cdot s^{-3}|$. Applying Weyl integral formula for G and G^*, we get the following character formulae.

Proposition. *Characters of induced representations defined above exist. The characters of $\rho(\pi, \eta)$ and $\rho^*(r, \eta)$ are given, respectively, by $\mathcal{X}(\pi, \eta)$ and $\mathcal{X}^*(r, \eta)$ defined as follows:*

$$\mathcal{X}(\pi, \eta)(g)=\begin{cases}\dfrac{1}{2}\Big[\sum_{W_1}\mathcal{X}_\pi\Big(\begin{pmatrix}a & \\ & b\end{pmatrix}\Big)d^{1/2}\Big(\begin{pmatrix}a & \\ & b\end{pmatrix}\Big)\eta(s)\Big]D(g)^{-1/2} \\ \qquad \text{if } g\approx t_1=\begin{pmatrix}a & & & \\ & b & & \\ & & sa^{-1} & \\ & & & sb^{-1}\end{pmatrix}\in T_1(F)^{\mathrm{reg}},\end{cases}$$

$$\begin{cases} \dfrac{1}{2}\Big[\sum_{W_{3,E}} \mathcal{X}_\pi(A)d^{1/2}(A)\eta(s)\Big]D(g)^{-1/2} \\ \qquad \text{if } g \approx t_3=\begin{pmatrix} A & \\ & s^tA^{-1}\end{pmatrix} \in T_{3,E}(F)^{\text{reg}}, \\ 0 \quad \text{otherwise.} \end{cases}$$

Here $T_1(F)^{\text{reg}}$ is the set of regular elements of $T_1(F)$, etc. And '$\approx$' means to 'be G-conjugate to' and the summation is over all the conjugates of t_1 [or t_3] by the action of W_1 [or $W_{3,E}$].

$$\mathcal{X}^*(r,\eta)(g)=\begin{cases} \dfrac{1}{2}\Big[\sum_{W_{3,E}} \mathcal{X}_r(\alpha)d^{*-1/2}(\alpha)\eta(s)\Big]D^{*-1/2}(g) \\ \qquad \text{if } g\approx t_3^*=\begin{pmatrix} \alpha & \\ & s\bar{\alpha}^{-1}\end{pmatrix} \in T^*_{3,E}(F)^{\text{reg}}, \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\mathcal{X}_\pi$ [or $\mathcal{X}_r$] is the character of $\pi \in R(GL(2,F))$ [or $r \in R(D^\times)$] and where d^* and D^* are defined in the same way as d and D, respectively.

Now let $\rho^*(r,\eta)$ correspond to $\rho(\pi(r),\eta)$ where $r \in R(D^\times) \mapsto \pi(r) \in R(GL(2,F))$ is the correspondence defined in Jacquet and Langlands [3]. Then we get the following character relation.

Theorem. *Let T and T^* be the corresponding F-tori of G and G^*. Then the following character relation*

$$\mathcal{X}^*(r,\eta)|_{T^*(F)}+\mathcal{X}(\pi(r),\eta)|_{T(F)}=0$$

holds independently of the choice of an F-isomorphism of $T^*(F)$ into $T(F)$. Here $|_{T^*(F)}$ [or $|_{T(F)}$] means restriction of $\mathcal{X}^*(r,\eta)$ [or $\mathcal{X}(\pi(r),\eta)$] to $T^*(F)$ [or $T(F)$]. Moreover the central character of $\rho^*(r,\eta)$ and that of $\rho(\pi(r),\eta)$ are the same.

The correspondences of representations of G and G^* parametrized by the duals of maximal F-tori of types (4) and (5) are not yet known. In order to find them, it is necessary to construct irreducible absolutely cuspidal representations of G and G^*, and to calculate their characters. We hope to discuss the subject in detail in near future.

References

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