55. The Exponential Calculus of Microdifferential Operators of Infinite Order. IV

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(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1983)

1. Introduction. In this note we show that there is a formal symbol q of order at most $1-0$ (see [1], [2] for the notation) satisfying

(1.1) $$\exp : p := : \exp q :.$$

Here p is a formal symbol of order at most $1-0$. That is, the exponential of an operator has an exponential symbol. Such q can be calculated from p.

2. Exponential of operators. Let X be an open set in C^n with coordinates $x=(x_1, \cdots, x_n)$, Ω a conic open set in $T^*X \simeq X \times C^n_\xi$. Let $p(t; x, \xi)$ be a formal symbol of order at most $1-0$ defined in Ω. We shall consider the operator $\exp(s : p(t; x, \xi) :)$ $(s \in C)$. Let us define a sequence $\{p^{(k)}(t; x, \xi)\}$ $(k=0, 1, 2, \cdots)$ of formal symbols by

(2.1) $$p^{(0)}(t; x, \xi)=1,$$

(2.2) $$p^{(k+1)}(t; x, \xi)=\exp(t\partial_\xi \cdot \partial_y) p(t; x, \xi) p^{(k)}(t; y, \eta)|_{\eta=\xi}^{y=x}.$$

Here $k=0, 1, 2, \cdots$. Then we set

(2.3) $$P(t; s, x, \xi)=\sum_{k=0}^{\infty} \frac{s^k}{k!} p^{(k)}(t; x, \xi).$$

Here $s \in C$. By the definition we have $: p^{(k)}(t; x, \xi) := (: p(t; x, \xi) :)^k$. Therefore $P(t; s, x, \xi)$ formally satisfies the following differential equation:

(2.4) $$\partial_s P(t; s, x, \xi)=\exp(t\partial_\xi \cdot \partial_y) p(t; x, \xi) P(t; s, y, \eta)|_{\eta=\xi}^{y=x},$$

(2.5) $$P(t; 0, x, \xi)=1.$$

Moreover we have

Proposition 1. *For every $s \in C$ the formal power series $P(t; s, x, \xi)$ in t is a formal symbol defined in Ω.*

Hence $P(t; s, x, \xi)$ defines an operator $: P(t; s, x, \xi) :$ which satisfies

(2.6) $$\partial_s : P(t; s, x, \xi) := : p(t; x, \xi) :: P(t; s, x, \xi) :,$$

(2.7) $$: P(t; 0, x, \xi) :=1.$$

Therefore $\exp(s : p(t; x, \xi) :)$ makes sense, which is defined by

(2.8) $$\exp(s : p(t; x, \xi) :) = : P(t; s, x, \xi) :.$$

3. Statement of the results. Let Ω be a conic open set in T^*X, $p(t; x, \xi)=\sum_{j=0}^{\infty} t^j p_j(x, \xi)$ a formal symbol of order at most $1-0$ defined in Ω. Let us define two sequences of symbols $\{\psi_{l,k}^{(j)}(x, y, \xi, \eta)\}$ and $\{q_k^{(j)}(x, \xi)\}$ defined respectively in $\Omega \times \Omega$ and in Ω by the following recursion formulae:

(3.1) $\psi_{l,0}^{(0)}=p_l(x,\xi), \qquad l=0,1,2,\cdots,$

(3.2) $\psi_{l,0}^{(j)}=0, \quad j>0, \quad l=0,1,2,\cdots,$

(3.3) $$q_k^{(j+1)}(x,\xi)=\frac{1}{j+1}\sum_{l=0}^{k}\psi_{l,k-l}^{(j)}(x,x,\xi,\xi),$$

(3.4) $$\psi_{l,k+1}^{(j)}=\frac{1}{k+1}\left\{\partial_\xi\cdot\partial_y\psi_{l,k}^{(j)}+\sum_{\nu=0}^{l}\sum_{\mu=0}^{j-1}\partial_\xi\psi_{\nu,k}^{(\mu)}\cdot\partial_y q_{l-\nu}^{(j-\mu)}(y,\eta)\right\}.$$

If $\psi_{\mu,\nu}^{(i)}$ is known for $i+\mu+\nu\leq m-1$ then $q_k^{(j)}$ is defined for $k+j\leq m$ by (3.3). Then $\psi_{\mu,\nu}^{(i)}$ is determined for $i+\mu+\nu=m$ by (3.1), (3.2), (3.4). Now we define a formal power series in t by

(3.5) $$q(t;s,x,\xi)=\sum_{k=0}^{\infty}t^k\sum_{j=1}^{k+1}s^j q_k^{(j)}(x,\xi), \qquad s\in\boldsymbol{C}.$$

Then we have

Theorem 2. *The formal series $q(t;s,x,\xi)$ is a formal symbol of order at most $1-0$ defined in Ω so that*

(3.6) $$:\exp(q(t;s,x,\xi)):=\exp(s:p(t;x,\xi):)$$

holds in $\mathcal{E}^R$.

Let λ be a real number such that $0\leq\lambda<1$.

Theorem 3. *If $p_l(x,\xi)$ is of order at most $(l+1)\lambda-l$ for each $l=0,1,2,\cdots$, then $q_k^{(j)}(x,\xi)$ is of order at most $(k+1)\lambda-k$ for every $k=0,1,2,\cdots,1\leq j\leq k+1$. Hence $q_k(s,x,\xi)=\sum_{j=1}^{k+1}s^j q_k^{(j)}(x,\xi)$ is also of order at most $(k+1)\lambda-k$ for any k.*

The preceding theorem declares that $p(t;x,\xi)$ and $q(t;1,x,\xi)$ have the same principal part.

4. Outline of the proof of Theorem 2. We assume that $P(t;s,x,\xi)$ defined by (2.3) can be written in the form

(4.1) $$P(t;s,x,\xi)=\exp(q(t;s,x,\xi)).$$

Then the left-hand side of (2.4) is $\partial_s q(t;s,x,\xi)\exp(q(t;s,x,\xi))$. It follows from the result of our preceding note [2] that the right-hand side of (2.4) is written in the form

(4.2) $$\varphi(t;s,x,\xi)\exp(q(t;s,x,\xi)).$$

Here φ is a formal symbol of order at most $1-0$ that can be calculated from q. We set $q(t;s,x,\xi)=\sum s^j t^k q_k^{(j)}(x,\xi)$ and define $q_k^{(j)}$ successively from the following identity:

(4.3) $$\partial_s q(t;s,x,\xi)=\varphi(t;s,x,\xi).$$

Then we have (3.1)–(3.4). Detailed proof will be published elsewhere.

References

[1] T. Aoki: Calcul exponentiel des opérateurs microdifférentiels d'ordre infini, I (to appear in Ann. Inst. Fourier).

[2] ——: The exponential calculus of microdifferential operators of infinite order. III. Proc. Japan Acad., 59A, 79–82 (1983).