52. Group Factors of the Haagerup Type

By Marie CHODA
Department of Mathematics, Osaka Kyoiku University

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1. Let N be a type II_1 factor with the canonical trace τ. We call it a factor of the *Haagerup type* if there exists a net $(P_\alpha)_\alpha$ of normal linear maps on N which satisfy the following conditions;

(1) each P_α is completely positive on N,

(2) each P_α is compact (i.e. for any $\varepsilon>0$, there exists a finite dimensional linear map Q on N such that $\|P_\alpha(x)-Q(x)\|_2<\varepsilon\|x\|_2$ for all $x \in N$),

and

(3) $\|P_\alpha(x)-x\|_2 \to 0$, for all $x \in N$.

Here, we put $\|x\|_2=\tau(x^*x)^{1/2}$ for $x \in N$.

This is a factor in the "Haagerup case" following A. Connes, and he remarked that each subfactor of a factor of the Haagerup type is again of the Haagerup type ([4]). Hence in the set of full II_1 factors, the class of the Haagerup type constitutes a minimal class.

In this paper, we shall characterize a property of an ICC group G, that its group von Neumann algebra $R(G)$ is to be of the Haagerup type. We shall call this property of the group the *property* (H). In [1], Akemann and Walter have investigated relations among various properties of locally compact groups, and they showed, in particular, that a group G does not have the property (T) of Kazhdan if G has the property (H). Now an application of our characterization shows that a group G may not have the property (H), even if G does not have the property (T). Another conclusion is that the full II_1 factors $R(F_2)$, $R(SL(3,Z))$ and $R(F_2\times SL(3,Z))$ are not isomorphic.

2. Let G be a discrete countable group. We denote by λ the left regular representation of $G: (\lambda(g)\xi)(h)=\xi(g^{-1}h)$ $(g, h \in G,\ \xi \in l^2(G))$. The group von Neumann algebra $R(G)$ is the von Neumann algebra on the Hilbert space $l^2(G)$ which is generated by $\{\lambda(g)\,;\, g \in G\}$. The algebra $R(G)$ is a type II_1 factor if and only if G is an ICC group (i.e. the class $\{hgh^{-1}\,;\, h \in G\}$ is infinite for each $g \in G\backslash\{1\}$, where 1 is the identity of G). For a $g \in G$, let $\delta(g)$ be the characteristic function of $\{g\}$. Then the factor $R(G)$ has the unique trace τ defined by $\tau(x)=(x\delta(1), \delta(1))$ for all $x \in R(G)$. Each $x \in R(G)$ has a unique form $x=\sum_{g\in G} x(g)\lambda(g)$ ($x(g)$ is a scalar for all $g \in G$) in the sense of $\|\cdot\|_2$-metric convergence.

Definition. A countable infinite group G is said to have the *property* (*H*) if there exists a net $(\varphi_\alpha)_\alpha$ of functions on G which satisfy the following conditions;

(1′) each φ_α is positive definite,

(2′) each φ_α vanishes at infinity (i.e. for any $\varepsilon>0$, there exists a finite subset F of G such that $|\varphi_\alpha(g)|<\varepsilon$ for all $g \in G\backslash F$),

and

(3′) $\varphi_\alpha(g)\to 1$ for all $g \in G$.

A linear map P on a von Neumann algebra M is completely positive if for each integer n the operator $(P(x_{ij}))$ is positive for a positive operator (x_{ij}) in the n by n matrix algebra on M.

Lemma 1. *Let P be a linear map on $R(G)$ and φ be a function on G defined by*

$$\varphi(g)=\tau(P(\lambda(g))\lambda(g)^*), \qquad g \in G.$$

(i) *If P is completely positive, then φ is positive definite.*

(ii) *If P is compact, then φ vanishes at infinity.*

Proof. Take a finite subset $(g_i)_{i=1}^n$ in G and a set $(c_i)_{i=1}^n$ of complex numbers. Then

$$\begin{aligned}\textstyle\sum_{i,j} c_i\bar{c}_j\varphi(g_j^{-1}g_i) &= \textstyle\sum_{i,j} c_i\bar{c}_j\tau(P(\lambda(g_j^{-1}g_i)\lambda(g_i)^*\lambda(g_j)) \\ &= \textstyle\sum_{i,j} c_i\bar{c}_j\tau(\lambda(g_j)P(\lambda(g_j^{-1}g_i))\lambda(g_i)^*) \\ &= \textstyle\sum_{i,j} (P(\lambda(g_j^{-1}g_i)c_i\delta(g_i^{-1}),\ c_j\delta(g_j^{-1}))\geqq 0\end{aligned}$$

if P is completely positive.

Assume that P is compact. Then for any $\varepsilon>0$, there exists a finite dimensional linear map Q on $R(G)$ such that $\|P(x)-Q(x)\|_2 \leqq (\varepsilon\|x\|_2)/2$ for all $x \in R(G)$. Let $\{y_1, \cdots, y_m\}\subset R(G)$ span $Q(R(G))$. We may assume that $\tau(y_iy_j^*)=0$ $(i\neq j)$ and $\|y_i\|_2=1$ for all i. Then there exists a finite subset F of G such that $\sum_{g\notin F}|\tau(y_i\lambda(g)^*)|^2<(\varepsilon/2mc)^2$ for all i, when $c=\sup\{\|Q(x)\|_2/\|x\|_2;\ 0\neq x \in R(G)\}$. Hence, for any $\varepsilon>0$, we have a finite subset F of G which satisfies that

$$\begin{aligned}|\varphi(g)|&=|\tau(P(\lambda(g))\lambda(g)^*)| \\ &\leqq \|P(\lambda(g))-Q(\lambda(g))\|_2+|\tau(Q(\lambda(g))\lambda(g)^*)|<\varepsilon\end{aligned}$$

for all $g \notin F$.

Thus φ vanishes at infinity.

Lemma 2. *Let φ be a positive definite function on an* ICC *group G. Then there exists a completely positive normal linear map P on $R(G)$ such that*

$$P(x)=\textstyle\sum_{g\in G} x(g)\varphi(g)\lambda(g) \qquad \text{for an } x=\sum_{g\in G} x(g)\lambda(g) \in R(G).$$

If φ vanishes at infinity, then the map P is compact.

Proof. We shall define the map P by the same way as in [5, Lemma 1]. Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the cyclic representation of G induced by φ. For a basis $(e_i)_i$ of H_φ, put $a_i(g)=(\pi_\varphi(g)e_i, \xi_\varphi)$ for all $g \in G$. Then $a_i \in l^\infty(G)$ for all i, $\sum_i|a_i(g)|^2<+\infty$ for all $g \in G$ and $\sum_i a_ia_i^* =\varphi(1)1$ as a multiplication operator on $l^2(G)$. For each $x \in R(G)$, we associate a bounded operator $\sum_i a_i x a_i^*$ on $l^2(G)$, which we shall denote by $P(x)$. Then P is σ-weakly continuous and $P(\lambda(g))=\varphi(g)\lambda(g)$ for all $g\in G$. Hence $P(x)=\sum_{g\in G} x(g)\varphi(g)\lambda(g)\in R(G)$ for an $x=\sum_{g\in G} x(g)\lambda(g)\in R(G)$. By the definition, P is completely positive.

Assume that φ vanishes at infinity. Then for each natural number k, we have a finite subset F_k of G such that $|\varphi(g)|<1/k$ for all $g\in G\backslash F_k$. Put, for each k,

$$P_k(x)=\sum_{g\in F_k}\varphi(g)x(g)\lambda(g) \qquad \text{for } x=\sum_{g\in G} x(g)\lambda(g)\in R(G).$$

Then $(P_k)_k$ is a sequence of finite rank linear maps on $R(G)$. For each $x\in R(G)$,

$$\begin{aligned}\|P(x)-P_k(x)\|_2^2 &= \|\textstyle\sum_{g\notin F_k} x(g)\varphi(g)\lambda(g)\|_2^2\\ &= \textstyle\sum_{g\notin F_k}|x(g)|^2|\varphi(g)|^2\\ &\leqq (\textstyle\sum_{g\notin F_k}|x(g)|^2)/k^2\leqq \|x\|_2^2/k^2.\end{aligned}$$

Hence P is compact.

Theorem 3. *Let G be an* ICC *group. Then the group von Neumann algebra $R(G)$ is of the Haagerup type if and only if G has the property (H).*

Proof. Assume that $R(G)$ is of the Haagerup type. Then there is a net $(P_\alpha)_\alpha$ of normal linear maps on $R(G)$ which satisfy (1)–(3). Put for each α,

$$\varphi_\alpha(g)=\tau(P_\alpha(\lambda(g))\lambda(g)^*), \qquad g\in G.$$

Then for each $g\in G$

$$\begin{aligned}|\varphi_\alpha(g)-1| &= |\tau(P_\alpha(\lambda(g))\lambda(g)^*)-1|\\ &\leqq |\tau(P_\alpha(\lambda(g))-\lambda(g))|\\ &\leqq \|P_\alpha(\lambda(g))-\lambda(g)\|_2\to 0.\end{aligned}$$

On the other hand, by Lemma 1, each φ_α is a positive definite function on G which vanishes at infinity. Hence G has the property (H).

Conversely assume that G has the property (H). Let $(\varphi_\alpha)_\alpha$ be a net of functions on G which satisfy (1′), (2′) and (3′). For each α, we have, by Lemma 2, a completely positive compact linear map P_α on $R(G)$ such that $P_\alpha(\sum_{g\in G} x(g)\lambda(g))=\sum_{g\in G} x(g)\varphi_\alpha(g)\lambda(g)$. Take an $\varepsilon>0$ and an $x=\sum_{g\in G} x(g)\lambda(g)\in R(G)$. Then there exists a finite subset F of G such that $\|x-\sum_{g\in F} x(g)\lambda(g)\|_2^2<\varepsilon/3$. Denote $\sum_{g\in F} x(g)\lambda(g)$ by x_F. Since $|\varphi_\alpha(g)|\leqq\varphi_\alpha(1)$ for all $g\in G$, we have that

$$\|P_\alpha(x)-P_\alpha(x_F)\|_2=\varphi_\alpha(1)\,(\textstyle\sum_{g\notin F}|x(g)|^2)^{1/2}<\varphi_\alpha(1)\varepsilon/3.$$

Hence, for each $x\in R(G)$,

$$\|P_\alpha(x)-x\|_2\leqq(1+\varphi_\alpha(1))\,\|x-x_F\|_2+(\textstyle\sum_{g\in F}|\varphi_\alpha(g)-1|^2|x(g)|^2)^{1/2}<\varepsilon,$$

for sufficiently large α, by the assumption for the net $(\varphi_\alpha)_\alpha$.

Hence $R(G)$ is of the Haagerup type.

A type II_1 factor N is said to be *full* if the inner automorphism group Int (N) is a closed subgroup of the automorphism group Aut (N) ([2]).

Let F_2 be a free group with two generators a and b. Then, for each $\alpha>0$, the function $\varphi_\alpha(g)=e^{-\alpha|g|}$ on F_2 is positive definite by [5], where $|g|$ is the length of the word for a $g \in F_2$. Hence $R(F_2)$ is a full II_1 factor of the Haagerup type. Take a t in the Torus which is irrational (mod 2π). Let θ be an automorphism of $R(F_2)$ such that $\theta(\lambda(a))=t\lambda(a)$ and $\theta(\lambda(b))=t\lambda(b)$. Then θ^n is outer for all n and a subsequence of $(\theta^n)_n$ converges to the identity. Therefore $\mathrm{Int}\,(R(F_2))$ is not open.

Let Γ be an ICC group with Kazhdan's property (T) (for example, $SL(3, Z)$). Contrary to $R(F_2)$, $\mathrm{Int}\,(R(\Gamma))$ is open ([3]).

Next corollary shows that $\{R(F_2), R(F_2)\otimes R(\Gamma), R(\Gamma)\}$ is a triple of non-isomorphic full II_1 factors.

Corollary 4. *The direct product of $F_2 \times \Gamma$ of F_2 and Γ is an* ICC *group which has neither the property (T) nor the property (H).*

The tensor product $R(F_2)\otimes R(\Gamma)$ of $R(F_2)$ and $R(\Gamma)$ is a full II_1 *factor which is not of the Haagerup type and* $\mathrm{Int}\,(R(F_2)\otimes R(\Gamma))$ *is not open.*

Proof. Since a subsequence of outer automorphisms $((\theta\otimes 1)^n)$ on $R(F_2)\otimes R(\Gamma)$ converges to the identity, $\mathrm{Int}\,(R(F_2)\otimes R(\Gamma))$ is not open. Hence the group $F_2\times\Gamma$ does not have the property (T) ([3]). Assume that $F_2\times\Gamma$ has the property (H). Then there exists a net $(\varphi_\alpha)_\alpha$ of functions on $F_2\times\Gamma$ which satisfy (1′), (2′) and (3′). By restricting each φ_α on $\{1\}\times\Gamma$, we would see that Γ has the property (H). This is a contradiction. Hence $F_2\times\Gamma$ does not have the property (H), so that $R(F_2)\otimes R(\Gamma)$ is not of the Haagerup type by Theorem 3.

References

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