51. On a Multi-Dimensional $[\alpha, \beta, \gamma]$-Langevin Equation

By Yuji NAKANO*) and Yasunori OKABE**)

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§ 1. **Introduction.** In this note we treat a d-dimensional stationary Gaussian process $X=(X(t)\,;\, t\in \boldsymbol{R})$ which satisfies the following stochastic differential equation, $[\alpha, \beta, \gamma]$-langevin equation

$$(1.1)\quad dX(t)=\left(-\beta X(t)+\int_{-\infty}^{0}\gamma(s)X(t+s)ds\right)dt+\alpha dB(t),$$

where (i) α and β are symmetric positive definite $d\times d$-matrices (ii) γ is a $d\times d$-matrix valued L^1-function on $(-\infty, 0)$ (iii) $(B(t)\,;\, t\in\boldsymbol{R})$ is a d-dimensional Brownian motion having the causal condition: $\sigma(X(s)\,;\, s\in(-\infty, t])=\sigma(B(s_1)-B(s_2)\,;\, s_1, s_2\in(-\infty, t])$ for any $t\in\boldsymbol{R}$.

The purpose of this note is to investigate under what condition the solution X of equation (1.1) has T-positivity. By T-positivity we mean that the covariance function R of X can be represented in the form

$$(1.2)\quad R(t)=\int_{[0,\infty)}e^{-|t|\lambda}\sigma(d\lambda)\qquad (t\in\boldsymbol{R}),$$

where σ is a bounded $d\times d$-Borel measure matrix valued function on $[0, \infty)$. One of the authors ([1]) has shown that a one-dimensional stationary Gaussian process X has T-positivity with $\sigma(\{0\})=0$ and $\int_0^\infty(\lambda^2+\lambda^{-1})\sigma(d\lambda)<\infty$ if and only if X satisfies $[\alpha, \beta, \gamma]$-Langevin equation with a triple $[\alpha, \beta, \gamma]$ satisfying

(1.3) α and β are positive numbers

(1.4) $\gamma(s)=\int_{[0,\infty)}e^{s\lambda}\mu(d\lambda)$ with a Borel measure μ on $[0, \infty)$ satisfying the conditions $\mu(\{0\})=0$ and $\beta>\int_{[0,\infty)}\lambda^{-1}\mu(d\lambda)$.

Taking into account of (1.3) and (1.4), we are now given a triple $[\alpha, \beta, \gamma]$ such that

(1.5) α and β are symmetric positive definite $d\times d$-matrices

(1.6) $\gamma(s)=\sum_{n=1}^{N}\mu^{(n)}e^{sq_n}$ with non-negative definite matrices $\mu^{(n)}$ $(1\leq n\leq N)$ and distinct positive numbers q_n $(1\leq n\leq N)$

*) Department of Economics, Shiga University, 1-1, Banba 1-chome, Hikone, Shiga 522.

**) Department of Mathematics, Faculty of Science, University of Tokyo, Hongo, Tokyo 113.

(1.7) $\beta-\sum_{n=1}^{N}\frac{\mu^{(n)}}{q_n}$ is a positive definite $d\times d$-matrix.

At first we treat the case $d=2$ and $\alpha=I$. Then we have the following key lemma.

Key lemma. *There exist a natural integer M, positive numbers p_n $(1\leq n\leq M)$ and non-negative definite 2×2-matrices $K_n(1\leq n\leq M)$ such that*

$$(1.8)\quad \left(\beta-(i\zeta)I-\int_{-\infty}^{0}e^{-i\zeta s}\gamma(ds)\right)^{-1}=\sum_{n=1}^{M}\frac{K_n}{p_n-i\zeta}\qquad (\zeta\in C^{+}).$$

By virtue of this Key lemma, we get the following

Theorem 1.1. (i) *There exist a pair of 2-dimensional stationary Gaussian process X and 2-dimensional Brownian motion $\boldsymbol{B}$ which satisfies $[\alpha,\beta,\gamma]$-Langevin equation* (1.1).

(ii) *X has T-positivity if and only if*

$$(1.9)\quad [\beta,\mu^{(n)}]=\sum_{m=1}^{N}\frac{[\mu^{(m)},\mu^{(n)}]}{q_m+q_n}\qquad \textit{for any } n\in\{1,2,\cdots,N\}.$$

Conversely, let X be any d-dimensional stationary Gaussian process having T-positivity with its covariance function R of the form (1.2) such that $\sigma=\sum_{n=1}^{M}\sigma^{(n)}\delta_{\{p_n\}}$ with positive definite $d\times d$-matrices $\sigma^{(n)}$ $(1\leq n\leq M)$ and distinct positive numbers p_n $(1\leq n\leq M)$. Then we have

Theorem 1.2. (i) *If $\sigma^{(n)}$ $(1\leq n\leq M)$ commute mutually, then there exists a triple $[\alpha,\beta,\gamma]$ satisfying* (1.5)–(1.7) *such that X satisfies $[\alpha,\beta,\gamma]$-Langevin equation* (1.1).

(ii) *If $M\leq 3$, then the necessary and sufficient condition that X satisfies $[\alpha,\beta,\gamma]$-Langevin equation with some triple $[\alpha,\beta,\gamma]$ satisfying* (1.5)–(1.7) *and is that $\sigma^{(n)}$ $(1\leq n\leq M)$ commute mutually.*

Finally we can get a generalized Einstein relation for the solution X of $[\alpha,\beta,\gamma]$-Langevin equation (1.1).

Theorem 1.3. $\alpha^2/2=R(0)C_{\beta,\gamma}$,

where

$$C_{\beta,\gamma}=\pi\left(\int_{R}((\alpha\beta\alpha^{-1}-i\xi I+\alpha\hat{\gamma}(\xi)\alpha^{-1})^{-1}(\alpha\beta\alpha^{-1}+i\xi I+\alpha\hat{\gamma}(-\xi)\alpha^{-1})^{-1})d\xi\right)^{-1}.$$

and the symbol ^ denotes Fourier transform.

§ 2. Outline of proofs. For the proof of Key lemma we define a 2×2-matrix valued function Z on C^{+} by

$$(2.1)\quad Z(\zeta)=\beta+i\zeta I-\int_{-\infty}^{0}e^{-i\zeta s}\gamma(ds).$$

Without loss of generality, we can assume that $q_n<q_{n+1}$ $(1\leq n\leq N-1)$ and $\mu^{(n)}$ $(1\leq n\leq N)$ are all positive definite. From ([1]) there exist positive numbers a_n and b_n $(1\leq n\leq N+1)$ such that (i) $a_n<q_n<a_{n+1}$ and $b_n<q_n<b_{n+1}$ $(1\leq n\leq N)$

(ii) $Z_{11}(\zeta)=\dfrac{\prod_{n=1}^{N+1}(-i\zeta+a_n)}{\prod_{n=1}^{N}(-i\zeta+q_n)}$ and $Z_{22}(\zeta)=\dfrac{\prod_{n=1}^{N+1}(-i\zeta+b_n)}{\prod_{n=1}^{N}(-i\zeta+q_n)}$.

We set $Z_0(\zeta)=Z(-i\zeta)$. Then we see that for any $n\in\{1, 2, \cdots, N\}$ $\det Z_0$ is continuous in $(-q_n, -q_{n+1})$, $\det Z_0(-q_n+0)=\det Z_0(-q_n-0)=\infty$ and further for any $n\in\{1, 2, \cdots, N+1\}$ $\det Z_0(-a_n)\leq 0$ and $\det Z_0(-b_n)\geq 0$, which implies that there exist $2N+2$ positive numbers r_n such that (i) $r_{2n-1}\leq r_{2n}<q_n<q_{2n+1}\leq r_{2n+2}$ $(1\leq n\leq N)$ (ii) the set of zeros of $\det Z_0=\{-ir_n\,;\,1\leq n\leq 2N+2\}$. It then follows that there exist an integer M, positive numbers p_n $(1\leq n\leq M)$ and 2×2-matrices K_n $(1\leq n\leq M)$ such that

(i) $p_n<p_{n+1}$ $(1\leq n\leq M-1)$

(ii) $\sum_{n=1}^{M} K_n=I$

(iii) $(Z(\zeta))^{-1}=\sum_{n=1}^{M}\dfrac{K_n}{-i\zeta+p_n}$.

It can be readily verified that K_n $(1\leq n\leq M)$ are non-negative definite. Therefore we have completed the proof of Key lemma.

By virtue of Key lemma, similarly to Theorem 3.1 and Lemma 4.1 in [1], we can get Theorem 1.1 (i). By calculating the covariance function R, we find that $R(t)=\sum_{n=1}^{M} e^{-p_n|t|}\sigma^{(n)}$ for $t>0$ and $R(t)=\sum_{n=1}^{M} e^{-p_n|t|}(\sigma^{(n)})^*$ for $t<0$, where $(\sigma^{(n)})^*$ denotes the transpose of $\sigma^{(n)}$ and

$$\sigma^{(n)}=\sum_{n=1}^{M}\frac{K_nK_m}{p_n+p_m}\qquad(1\leq n\leq M).$$

A direct calculation gives Theorem 1.1 (ii).

By diagonizing the matrices $\sigma^{(n)}$ $(1\leq n\leq M)$ simultaneously, we find from Theorem 6.1 in [1] that Theorem 1.2 (i) holds. By noting that if $M\leq 3$, then the matrices K_n $(1\leq n\leq M)$ commute mutually, we can get Theorem 1.2 (ii).

Theorem 1.3 follows from the same consideration as (9.13) in [1].

Reference

[1] Y. Okabe: On a stochastic differential equation for a stationary Gaussian process with T-positivity and fluctuation-dissipation theorem. J. Fac. Sci. Univ. of Tokyo, Sec. 1A, **28**, 169–213 (1981).