50. Toda Lattice Hierarchy. I

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0. Introduction. In this note we shall consider a hierarchy (a sequence of mutually commuting higher evolutions) for the two dimensional Toda lattice (TL)

(1) $$\partial_{x_1}\partial_{y_1}u(s)=e^{u(s)-u(s-1)}-e^{u(s+1)-u(s)},$$

where $u(s)=u(s;x_1,y_1)$, $\partial_{x_1}=\partial/\partial x_1$, $\partial_{y_1}=\partial/\partial y_1$ and s runs over Z, the totality of integers. The Toda lattice is, together with the Korteweg-de Vries (KdV) equation, one of the most classical and important completely integrable systems. Several varieties of methods have been developed to reveal its profound mathematical structure.

The present work is inspired by the recent developments in the study of the Kadomtsev-Petviashvili (KP) equation and its hierarchies [1], [2]. Our method enables us to investigate the infinite Toda lattice in an extremely clear and algebraic framework.

Let us begin with the following observations: In the periodic case the so called zero curvature representation of (1) is given in [3]. In the general case (1) is represented in the form

(2) $$\partial_{y_1}B_1-\partial_{x_1}C_1+[B_1,C_1]=0,$$

where the symbol [,] denotes the commutator, and B_1, C_1 are the matrices $B_1=(\delta_{i,j-1})_{ij\in Z}+(\partial_{x_1}u(i)\delta_{i,j})_{ij\in Z}$, $C_1=(e^{u(i)-u(i-1)}\delta_{i,j+1})_{ij\in Z}$ of size $Z\times Z$.

If the τ function $\tau(s)=\tau(s;x_1,y_1)$ is introduced by

(3) $$u(s)=\log[\tau(s+1)/\tau(s)],$$

(1) is transformed into the bilinear equation of the Hirota type

(4) $$D_{x_1}D_{y_1}\tau(s)\cdot\tau(s)+2\tau(s+1)\tau(s-1)=0,$$

where $D_{x_1}D_{y_1}$ is one of Hirota's D-operators which are in general defined, for a linear differential operator $P(\partial_t)$, by

(5) $$P(D_t)f(t)\cdot g(t)=P(\partial_{t'})f(t+t')g(t-t')|_{t'=0}.$$

By use of the function $\tau'(s)=e^{x_1y_1}\tau(s)$, (4) is rewritten into the original form of Hirota's bilinearization [4]

(6) $$D_{x_1}D_{y_1}\tau'(s)\cdot\tau'(s)+2\tau'(s+1)\tau'(s-1)-2\tau'(s)^2=0.$$

The N soliton solution to (6) is presented in [4]. A parametrization of the τ function $\tau'(s)$ in terms of Clifford operators is discussed in [5].

We shall extend these observations to a hierarchy for (1).

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1. Some notations for matrices of size $Z\times Z$. A matrix A of size $Z\times Z$ is expressed in the form

$$A=\sum_{n\in Z}\operatorname{diag}[a_n(s)]\Lambda^n,$$

where Λ^n denotes the n-th shift matrix $\Lambda^n=(\delta_{i,j-n})_{ij\in Z}$ and $\operatorname{diag}[a_n(s)]$ the diagonal matrix $(a_n(i)\delta_{i,j})_{ij\in Z}$. $(A)_\pm$ is defined by

$$(A)_+=\sum_{n\ge 0}\operatorname{diag}[a_n(s)]\Lambda^n,\qquad (A)_-=\sum_{n<0}\operatorname{diag}[a_n(s)]\Lambda^n.$$

We say that A is of finite width if there is a constant m such that $a_n(s)$ vanishes for $|n|>m$.

2. Formulation of the hierarchy. Let $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ be independent variables with infinitely many components. Let us introduce matrices L, M, B_n, C_n, $n=1, 2, \cdots$, of size $Z\times Z$ of the form

$$(7)\qquad \begin{aligned} L&=\sum_{-\infty<n\le 1}\operatorname{diag}[b_n(s\,;x, y)]\Lambda^n, && b_1(s\,;x,y)=1,\\ M&=\sum_{-1\le n<\infty}\operatorname{diag}[c_n(s\,;x, y)]\Lambda^n, && c_{-1}(s\,;x,y)\neq 0,\end{aligned}$$

$$(8)\qquad \begin{aligned} B_n&=(L^n)_+\Big(=\sum_{m=0}^{n}\operatorname{diag}[b_{n,m}(s\,;x,y)]\Lambda^{n-m}\Big),\\ C_n&=(M^n)_-\Big(=\sum_{m=0}^{n-1}\operatorname{diag}[c_{n,m}(s\,;x,y)]\Lambda^{n-m}\Big).\end{aligned}$$

b_j $(j\le 1)$ and c_j $(j\ge -1)$ serve as the unknown functions in the hierarchy. $b_{n,m}$ and $c_{n,m}$ are expressed as polynomials of b_j $(j\le 1)$, c_j $(j\ge -1)$ and their shifts in the variable s.

Our Toda lattice hierarchy (TL hierarchy) is, by definition, the following system of the Lax type

$$(9)\qquad \begin{aligned} &\partial_{x_n}L=[B_n, L], && \partial_{y_n}L=[C_n, L],\\ &\partial_{x_n}M=[B_n, M], && \partial_{y_n}M=[C_n, M], \quad n=1,2,\cdots.\end{aligned}$$

Another representation of the TL hierarchy is presented in

Theorem 1. (9) *is equivalent to the system of the Zakharov-Shabat type*

$$(10)\qquad \begin{aligned} &\partial_{x_n}B_m-\partial_{x_m}B_n+[B_m, B_n]=0, && \partial_{y_n}C_m-\partial_{y_m}C_n+[C_m, C_n]=0,\\ &\partial_{y_n}B_m-\partial_{x_m}C_n+[B_m, C_n]=0, && m, n=1,2,\cdots.\end{aligned}$$

Notice that (2) *is contained in* (10).

3. Linearization. The linearization of the TL hierarchy is achieved by the following linear system for the unknown matrices $W^{(\infty)}$, $W^{(0)}$ of size $Z\times Z$,

$$(11)\qquad \begin{aligned} &LW^{(\infty)}=W^{(\infty)}\Lambda, \qquad MW^{(0)}=W^{(0)}\Lambda^{-1},\\ &\partial_{x_n}W=B_nW, \quad \partial_{y_n}W=C_nW, \quad W=W^{(\infty)}, W^{(0)}, \quad n=1,2,\cdots,\end{aligned}$$

where $W^{(\infty)}$ and $W^{(0)}$ are assumed to have the following form,

$$W^{(\infty)}=\hat{W}^{(\infty)}e^{\xi(x,\Lambda)},\qquad W^{(0)}=\hat{W}^{(0)}e^{\xi(y,\Lambda^{-1})},$$

$$\hat{W}^{(\infty)}=\sum_{n=0}^{\infty}\mathrm{diag}\,[\hat{w}_n^{(\infty)}(s\,;x,y)]\Lambda^{-n},\qquad \hat{w}_0^{(\infty)}(s\,;x,y)=1,$$

$$\hat{W}^{(0)}=\sum_{n=0}^{\infty}\mathrm{diag}\,[\hat{w}_n^{(0)}(s\,;x,y)]\Lambda^{n},\qquad \hat{w}_0^{(0)}(s\,;x,y)\neq 0, \tag{12}$$

$$\xi(x,\Lambda)=\sum_{n=1}^{\infty}x_n\Lambda^n,\qquad \xi(y,\Lambda^{-1})=\sum_{n=1}^{\infty}y_n\Lambda^{-n}.$$

To be more precise, we have

Theorem 2. *If (9), (10) are satisfied there exist solutions $W^{(\infty)}$, $W^{(0)}$ to (11) of the form as (12), and they are unique up to the arbitrariness $\hat{W}^{(\infty)}\to\hat{W}^{(\infty)}\sum_{n=0}^{\infty}f_n\Lambda^{-n}$, $\hat{W}^{(0)}\to\hat{W}^{(0)}\sum_{n=0}^{\infty}g_n\Lambda^n$ (f_n, $g_n\in \boldsymbol{C}$, $f_0=1$, $g_0\neq 0$). Conversely if $W^{(\infty)}$, $W^{(0)}$ solve (11) for certain matrices L, M* (of the form as (7)) *and B_n, C_n* (of finite width), *then* (8), (9) *and* (10) *are satisfied.*

Notice that the first line of (11) implies that L, M are recovered from $W^{(\infty)}$, $W^{(0)}$ by

$$L=\hat{W}^{(\infty)}\Lambda\hat{W}^{(\infty)-1},\qquad M=\hat{W}^{(0)}\Lambda^{-1}\hat{W}^{(0)-1}. \tag{13}$$

Let us call the solutions $W^{(\infty)}$, $W^{(0)}$ to (11) the wave matrices for the TL hierarchy (as analogues of the wave functions in the classical inverse scattering theory). They have another characterization:

Theorem 3. *Matrices $W^{(\infty)}$ and $W^{(0)}$ of the form as* (12) *are wave matrices for the TL hierarchy iff the bilinear equation*

$$W^{(\infty)}(x',y')W^{(\infty)}(x,y)^{-1}=W^{(0)}(x',y')W^{(0)}(x,y)^{-1} \tag{14}$$

is satified for any x, y, x' and y'.

The linearization (11) can be rewritten in a more classical form: Introduce the following formal power series of λ (cf. (12)).

$$w^{(\infty)}(s\,;x,y\,;\lambda)=\hat{w}^{(\infty)}(s\,;x,y\,;\lambda)\lambda^s e^{\xi(x,\lambda)},$$

$$w^{(0)}(s\,;x,y\,;\lambda)=\hat{w}^{(0)}(s\,;x,y\,;\lambda)\lambda^s e^{\xi(y,\lambda^{-1})},$$

$$\hat{w}^{(\infty)}(s\,;x,y\,;\lambda)=\sum_{n=0}^{\infty}\hat{w}_n^{(\infty)}(s\,;x,y)\lambda^{-n}, \tag{15}$$

$$\hat{w}^{(0)}(s\,;x,y\,;\lambda)=\sum_{n=0}^{\infty}\hat{w}_n^{(0)}(s\,;x,y)\lambda^{n}.$$

The second line of (11) is then equivalent to the linear system (cf. (8))

$$\partial_{x_n}w^{(\infty)}(s\,;x,y\,;\lambda)=\sum_{m=0}^{n}b_{n,m}(s\,;x,y)w^{(\infty)}(s+n-m\,;x,y\,;\lambda),$$

$$\partial_{y_n}w^{(0)}(s\,;x,y\,;\lambda)=\sum_{m=0}^{n-1}c_{n,m}(s\,;x,y)w^{(0)}(s-n+m\,;x,y\,;\lambda), \tag{16}$$

$$n=1,2,\cdots.$$

4. Bilinearization. The bilinearization of the TL hierarchy is achieved by use of the τ function: Let $W^{(\infty)}$, $W^{(0)}$ be wave matrices of the TL hierarchy and introduce $\hat{w}^{(\infty)}$, $\hat{w}^{(0)}$ as in (15).

Theorem 4. *The τ function $\tau(s)=\tau(s\,;x,y)$ is consistently, and uniquely up to constant multipliers, introduced by the equations*

$$\hat{w}^{(\infty)}(s\,;x,y\,;\lambda)=\tau(s\,;x-\varepsilon(\lambda^{-1}),y)/\tau(s\,;x,y),$$

$$\hat{w}^{(0)}(s\,;x,y\,;\lambda)=\tau(s+1\,;x,y-\varepsilon(\lambda))/\tau(s\,;x,y), \tag{17}$$

where $\varepsilon(\lambda^{\pm 1})=(\lambda^{\pm 1}, \lambda^{\pm 2}/2, \lambda^{\pm 3}/3, \cdots)$.

It should be noticed that the transformation

$$\tau(s)\rightarrow\tau(s)g_0^s \exp\left(\sum_{n=1}^{\infty} d_n x_n+\sum_{n=1}^{\infty} e_n y_n\right) \qquad (d_n, e_n \in \boldsymbol{C})$$

exactly corresponds to the arbitrariness of the wave matrices mentioned in Theorem 2 with $\sum_{n=0}^{\infty} f_n \lambda^{-n}=\exp(-\sum_{n=1}^{\infty} d_n \lambda^{-n}/n)$, $\sum_{n=0}^{\infty} g_n \lambda^n =g_0 \exp(-\sum_{n=1}^{\infty} e_n \lambda^n/n)$.

Theorem 5. *The* τ *function satisfies* (and also is characterized by) *the bilinear equations*

$$\sum_{n=0}^{\infty} p_{n+m}(-2u)p_n(\tilde{D}_x)e^{\langle u,D_x\rangle+\langle v,D_y\rangle}\tau(s+m+1)\cdot\tau(s)$$

$$(18) \qquad =\sum_{n=0}^{\infty} p_{n-m}(-2v)p_n(\tilde{D}_y)e^{\langle u,D_x\rangle+\langle v,D_y\rangle}\tau(s+m)\cdot\tau(s+1), \qquad m\in\boldsymbol{Z},$$

where $\tilde{D}_x=(D_{x_1}, D_{x_2}/2, D_{x_3}/3, \cdots)$, $\langle u, D_x\rangle=\sum_{n=1}^{\infty} u_n D_{x_n}$, *and* p_n *is the polynomial defined by the generating function* $e^{\xi(x,\lambda)}=\sum_{n=0}^{\infty} p_n(x)\lambda^n$ [1], *while* $u=(u_1, u_2, \cdots)$ *and* $v=(v_1, v_2, \cdots)$ *are indeterminate variables.*

(18) is understood to be a generating functional expression of infinitely many bilinear equations of the Hirota type. They contain (4) as a special one, and also the bilinear equations for the KP hierarchy [2] which imply that $\tau(s; x, y)$ is, regarded as a function of x or y separately, a τ function for the KP hierarchy.

(18) can be transformed into bilinear equations for the function $\tau'(s; x, y)=\exp(\sum_{n=1}^{\infty} nx_n y_n)\tau(s; x, y)$ which consequently solves (6). $\tau'(s)$ is effectively used in discussing some classes of special solutions.

A parametrization of the τ functions in terms of the vacuum expectation values for Clifford operators is obtained as follows: Let $\tau_{l_1,l_2}(x^{(1)}, x^{(2)})$ $(l_1+l_2=0)$ be the vacuum expectation value introduced in [2, III], which was used to parametrize the τ functions for the two component KP hierarchy.

Theorem 6. *Suppose* $\tau_{s,-s}(x^{(1)}, x^{(2)})\not\equiv 0$ *for any* $s\in\boldsymbol{Z}$. *Then*

$$(19) \qquad \tau(s; x, y)=(-1)^{s(s-1)/2}\tau_{s,-s}(x, y)$$

presents a τ *function for the TL hierarchy.*

References

[1] M. Sato: Kokyuroku RIMS, Kyoto Univ., no. 439, pp. 30–46 (1981).

[2] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa: Proc. Japan Acad., **57**A, 342–347 (1981); ibid., 387–392; J. Phys. Soc. Japan, **40**, 3806–3812 (1981); ibid., 3813–3818; Physica, 4D, 343 (1982); RIMS Preprint, Kyoto Univ., no. 359 (1981).

[3] A. V. Mikhailov, M. A. Loshanetsky, and A. M. Perelomov: Comm. Math. Phys., **79**, 473–488 (1981).

[4] R. Hirota: J. Phys. Soc. Japan, **50**, 3875 (1981); Tech. Rep., A-6, A-9, Hiroshima Univ. (1981).

[5] T. Miwa: Proc. Japan Acad., **58**A, 8–11 (1982).