47. C^2 Reeb Stability of Noncompact Leaves of Foliations

By Takashi INABA
College of Arts and Sciences, Chiba University

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1. Introduction. The purpose of this note is to announce a result on the stability of noncompact leaves of codimension one foliations which extends a 2-dimensional theorem of Cantwell-Conlon [1] to all dimensions. We assume throughout that foliations are always transversely orientable, codimension one foliations of closed manifolds with smooth leaves. Recall that a proper leaf of a foliation is *stable* if it admits a trivially foliated, saturated neighborhood (see [3]).

Definition ([1]). A smooth manifold L has the *C^r-stability property* if, whenever L is diffeomorphic to a proper leaf of a C^r foliation, that leaf is stable.

The problem we consider is to characterize the stability property of a manifold L in terms of the topology of L. In the case when L is compact, Thurston [4] has almost completely settled this problem: a compact manifold L has the C^r-stability property $(1\leqq r\leqq\infty)$ if and only if $H^1(L;\boldsymbol{R})=0$. However, in the case when L is noncompact, few partial answers have been known. An important remark is that the direct analogue of Thurston's result does not hold in this case. In fact, it is shown in [1] that there are infinitely many noncompact surfaces with nontrivial real first cohomology groups which have the C^2-stability property. (Although they do not have the C^1-stability property.) Our results give a necessary condition (Proposition 1) and a sufficient condition (Theorem 3) under which a manifold has the C^2-stability property.

2. Statement of results. Let $\hat{H}^1(L;\boldsymbol{R})$ be the image of the canonical homomorphism $H_c^1(L;\boldsymbol{R})\to H^1(L;\boldsymbol{R})$, where H_c^1 denotes the first cohomology group with compact supports. (Note that $\hat{H}^1(L;\boldsymbol{R})$ coincides with $H^1(L;\boldsymbol{R})$ if L is compact.)

First we observe the following

Proposition 1. *Suppose that L is a manifold which can be realized as a proper leaf of some C^r foliation $(0\leqq r\leqq\infty)$. If L has the C^r-stability property, then $\hat{H}^1(L;\boldsymbol{R})=0$.*

This proposition says that the vanishing of $\hat{H}^1(L;\boldsymbol{R})$ is a necessary condition for the stability of L. It is, however, not a sufficient condition. In fact, for example, we have

Proposition 2. *$T^2\times\boldsymbol{R}$ does not have the C^r-stability property*

$(0 \leqq r \leqq \infty)$.

In order to obtain a sufficient condition, we make a certain restriction on the behavior of ends of manifolds. First we briefly recall some basic definitions about ends.

An end e is determined by a pair $(M, \{U_i\}_{i=0}^{\infty})$ where M is a manifold and $U_0 \supset U_1 \supset U_2 \supset \cdots$ is a decreasing sequence of nonempty, connected open subsets of M such that 1) $\bar{U}_i - U_i$ is compact for each i, and 2) $\bigcap_{i=0}^{\infty} \bar{U}_i = \phi$. Two pairs $(M, \{U_i\})$ and $(M', \{U'_i\})$ determine the same end if there exist a connected open subset W (resp. W') of M (resp. M') and a diffeomorphism $f: W \to W'$ such that 1) U_i (resp. U'_i) is contained in W (resp. W') for large i, and 2) every $f(U_i)$ contains some U'_j and every U'_i contains some $f(U_j)$. An end e is *periodic* if e is determined by $(M, \{g^i(U)\})$ where U is a subset of M and g a diffeomorphism of U *into* U. In this case, $\bar{U} - g(U)$ is called a *period* of e and U is called a *periodic neighborhood* of e.

Now we define a class $\mathcal{P}$ of ends as follows. $\mathcal{P}$ is described as a disjoint union of subsets $\mathcal{P}_k$ $(k=0, 1, 2, \cdots)$. An end e belongs to $\mathcal{P}_0$ if e is a periodic end of period $K \times I$, where K is a connected closed manifold satisfying the following condition:

(*) The quotient group of $\pi_1(K)$ by the smallest normal subgroup containing all torsion elements is isomorphic to $\{1\}$ or $\boldsymbol{Z}$.

Suppose we have defined $\mathcal{P}_i$ for $0 \leqq i \leqq k-1$. An end e belongs to $\mathcal{P}_k$ if e is constructed in the following way: Let K be a connected closed manifold satisfying (*), and let $B_1, B_2, \cdots, B_s$ be pairwise disjoint, codimension zero, compact submanifolds of $K \times \operatorname{Int} I$ such that ∂B_i satisfies (*) for each i. Let N_i $(1 \leqq i \leqq s)$ be a periodic neighborhood of an end $e_i \in \mathcal{P}_{k_i}$ $(0 \leqq k_i \leqq k-1$, and at least one of the k_i's is equal to $k-1)$ such that ∂N_i is diffeomorphic to ∂B_i. Then e is a periodic end of period $(K \times I - \bigcup_{i=1}^{s} \operatorname{Int} B_i) \cup_{\partial} (\bigcup_{i=1}^{s} N_i)$.

The main result of this note is the following

Theorem 3. *Let L be a smooth manifold such that $\hat{H}^1(L; \boldsymbol{R}) = 0$ and that all ends of L belong to $\mathcal{P}$. Then L has the C^r-stability property* $(2 \leqq r \leqq \infty)$.

Remarks. 1) If the condition (*) is dropped in the definition of $\mathcal{P}$, the conclusion of Theorem 3 does not hold (see Proposition 2).

2) Theorem 3 fails if $r \leqq 1$. (For example, the cylinder $S^1 \times \boldsymbol{R}$ does not have the C^1-stability property.)

3) If a proper leaf satisfies the hypothesis of Theorem 3, it actually has a saturated neighborhood which is fibered over S^1 with fibers as leaves.

3. Indication of proof. We use the following two theorems in the proof of Theorem 3.

Hector's uniform convergence theorem [2]. Let $\mathcal{F}$ be a C^2 foliation of a closed manifold M. Fix a "nice" finite cover of M by $\mathcal{F}$-charts. Let $\{\mathcal{P}_n^{-1}*Q_n*\mathcal{P}_n\}_{n\in N}$ be a sequence of basic cycles (=conjugates of simple plaque cycles Q_n by simple plaque chains $\mathcal{P}_n$) on a proper leaf of $\mathcal{F}$ based at a plaque P such that the distance between P and Q_n diverges to infinity as $n\to\infty$. Then the holonomy associated to $\mathcal{P}_n^{-1}*Q_n*\mathcal{P}_n$ converges to the identity uniformly.

Relative version of Thurston's generalized Reeb stability theorem [4]. Let K be a compact codimension zero submanifold of a C^1 foliation such that 1) each component of ∂K has trivial holonomy, and 2) the restriction homomorphism $i^*: H^1(K; \boldsymbol{R})\to H^1(\partial K; \boldsymbol{R})$ is injective. Then K is stable.

The proof of Theorem 3 proceeds roughly as follows: Let L be a proper leaf of a C^r foliation ($2\leqq r\leqq\infty$) satisfying the hypothesis of the theorem. By Hector's theorem and the condition on the ends of L, there is a compact subset K of L such that $L-K$ is stable. Then by Thurston's theorem and the condition that $\hat{H}^1(L; \boldsymbol{R})=0$, K is stable. Combining these, we see that L itself is stable.

Details will appear elsewhere in near future.

References

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[4] W. Thurston: A generalization of the Reeb stability theorem. ibid., **13**, 347–352 (1974).