

44. Vanishing Theorems in Asymptotic Analysis

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Let M be a complex manifold and let H be a divisor on M . Denote by \mathcal{O} the sheaf over M of germs of holomorphic functions and by $\mathcal{O}(*H)$ the sheaf over M of germs of meromorphic functions which are holomorphic in $M-H$ and have poles on H . We suppose that the divisor H has normal crossings. At a point p on H , we may choose local holomorphic coordinates (x_1, \dots, x_n) in a neighborhood $U = \prod_{i=1}^n \{|x_i| < r_i\}$ of $p = (0, \dots, 0)$ such that the set $H \cap U = \bigcup_{i=1}^{n''} \{x : x_i = 0\}$ is the union of coordinate hyperplanes. The complement $U - (U \cap H)$ is a punctured polycylinder $P^*(n'', n)$ given by

$$\begin{aligned} \{x : |x_i| < r_i \ (i=1, \dots, n), x_1, \dots, x_{n''} \neq 0\} \\ = \prod_{i=1}^{n''} (D(r_i) - \{0\}) \times \prod_{i=n''+1}^n D(r_i), \end{aligned}$$

where $D(r_i) = \{|x_i| < r_i\}$. Topologically, $P^*(n'', n)$ is a product $\times^{n''} S^1$ of n'' circles, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Let S be an open polysector at p of the form

$$S = \prod_{i=1}^{n''} S(c_i, r_i) \times \prod_{i=n''+1}^n D(r_i),$$

with the coordinates system x chosen as above, where we set

$$S(c_i, r_i) = \{x_i : 0 < |x_i| < r_i, \arg x_i \in c_i\}$$

for a positive number r_i and an open interval c_i in \mathbb{R} , $i=1, \dots, n''$, and let f be a holomorphic function in S . We denote by $[1, n]$ the set $\{1, 2, 3, \dots, n\}$. We say that f is *strongly asymptotically developable as x tends to H in S at p* , if there exists a family of functions

$$F = \{f(x_I; q_J) : \phi \neq J \subset [1, n''], I = [1, n] - J, q_J \in \mathbb{N}^J\}$$

satisfying the following properties:

(1.1) $f(x_I; q_J)$ is a holomorphic function of $x_I = (x_i)_{i \in I}$ in the open polysector

$$S_I = \prod_{i \in I \cap [1, n'']} S(r_i) \times \prod_{j=n''+1}^n D(r_j)$$

for any non-empty subset J of $[1, n'']$ and for any $q_J \in \mathbb{N}^J$,

(1.2) for any $N \in \mathbb{N}^{[1, n'']}$ and for any closed subpolysector

$$S' = \prod_{i=1}^{n''} S[c'_i, r'_i] \times \prod_{j=n''+1}^n D[r'_j]$$

of S , there exists a constant $K_{S', N}$ such that

$$|f(x) - \text{App}_N(x; F)| \leq K_{S', N} |x_{[1, n'']}|^N,$$

or any $x \in S'$, where we set

$$S[c'_i, r'_i] = \{x_i : 0 < |x_i| \leq r'_i, \arg x_i \in c'_i\}, D[r'_j] = \{x_j : |x_j| \leq r'_j\}$$

for a positive number $r'_i < r_i$ and a closed sub-interval c'_i of c_i ,

$i=1, \dots, n''$, for a positive real numbers $r'_j < r_j$, $j=n''+1, \dots, n$ and $\text{App}_N(x; F)$ is defined by

$$(1.3) \quad \text{App}_N(x; F) = \sum_{j \neq J \subset [1, n'']} (-1)^{\#J+1} \sum_{I \in J} \sum_{p_j=0}^{N_j-1} f(x_I; p_j) x_j^{p_j}.$$

If f is strongly asymptotically developable, the family of functions satisfying (1.1) and (1.2) is *uniquely* determined, which will be called the *total family of coefficients of strongly asymptotic expansion* of f and denoted by $TA(f)$. For a non-empty subset J of $[1, n'']$, we denote $f(x_I; q_J)$ by $TA(f)_{q_J}$ and define the formal series

$$(1.4) \quad FA_J(f) = \sum_{q_J \in N^J} TA_J(f)_{q_J} x_j^{q_j}$$

which will be called the *formal series of strongly asymptotic expansion* of $f(x)$ for $J \subset [1, n'']$. In the case in which $f(x_I; q_J)$ can be extended to a holomorphic function in $\prod_{i \in I} D(r_i)$, we say that f is *strongly asymptotically developable to the formal series*

$$\begin{aligned} f'(x_{[1, n'']}, x_{[n''+1, n]}) &= \sum_{q \in N^{n''}} f(x_{[n''+1, n]}, q)(x_{[1, n'']})^q \\ &= \sum_{q_j \in N} f(x_{[1, n]-(j)}; q_j)(x_j)^{q_j} (j \in [1, n'']), \end{aligned}$$

in $\mathcal{O}_{H \cap D(r)'} = \bigcap_{i=n''+1}^j \mathcal{O}(\prod_{i \neq j} D(r_i))[[x_j]]$ as x tends to H in S at p , where $\mathcal{O}(\prod_{i \neq j} D(r_i))[[x_j]]$ is the \mathbf{C} -algebra of formal series of x_j with coefficients in the \mathbf{C} -algebra $\mathcal{O}(\prod_{i \neq j} D(r_i))$ of holomorphic functions in $\prod_{i \neq j} D(r_i)$. Let \mathcal{J}_H be the nullstellen ideal of H and put $\mathcal{O}_M \hat{\wedge}_H = \text{proj. lim}_{k \rightarrow \infty} \mathcal{O}/(\mathcal{J}_H)^k$. Note that $\mathcal{O}_{H \cap D(r)'}$ coincides with the \mathbf{C} -algebra $\mathcal{O}_M \hat{\wedge}_H(D(r))$ of all sections of $\mathcal{O}_M \hat{\wedge}_H$ over $D(r)$.

Let U be an open set included in $M-H$ and including polysectors at any point in $\text{cl}(U) \cap H$, where $\text{cl}(U)$ denotes the closure of U in M . We say that a holomorphic function f in U is *strongly asymptotically developable as variables tend to H in U* , if, for any point $p \in \text{cl}(U) \cap H$ with holomorphic local coordinates system x and for any open polysector $S = \prod_{i=1}^{n''} S(c_i, r_i) \times \prod_{i=n''+1}^n D(r_i)$ included in U at p , f is strongly asymptotically developable as x tends to H in S at p . We denote by $\mathcal{A}(U)$ the set of all functions holomorphic in U and strongly asymptotically developable as the variables tend to H in U , and denote by $\mathcal{A}'(U)$ (or $\mathcal{A}_0(U)$) the subset of $\mathcal{A}(U)$ of functions strongly asymptotically developable to some formal series (or the identically zero series 0) as the variables tend to H in any open polysector at any point in $\text{cl}(U) \cap H$. The set $\mathcal{A}(U)$ has naturally a \mathbf{C} -algebraic structure, and $\mathcal{A}'(U)$ and $\mathcal{A}_0(U)$ are sub- \mathbf{C} -algebras of $\mathcal{A}(U)$: $\mathcal{A}_0(U)$ is also a sub- \mathbf{C} -algebra of $\mathcal{A}'(U)$. With these materials, we can define sheaves over the *real blow-up along H* , which is constructed in a similar manner as the construction of blow-up of a complex manifold along a submanifold as follows. We start by constructing the real blow-up of a polydisc along coordinate planes. Let $D(r)$ be an n -dimensional disc at the origin in \mathbf{C}^n with holomorphic coordinates x_1, \dots, x_n , and let $V \subset D(r)$ be the locus $\bigcup_{i=1}^{n''} \{x : x_i = 0\}$. Let $D(r)^-$ be the real analytic subvariety

of $D(r) \times (S^1)^{n''}$ defined by the relations

$$D(r)^- = \{(x_1, \dots, x_n, z_1, \dots, z_{n''}); \operatorname{Im}(x_i \bar{z}_i) = 0, \operatorname{Re}(x_i \bar{z}_i) \geq 0, i \in [1, n'']\}.$$

The projection $pr: D(r)^- \rightarrow D(r)$ on the first factor is clearly isomorphism away from V , while, for a point $x \in V$ such that $x_i = 0, i \in I$, and $x_j \neq 0, j \in [1, n''] - I$, where $I \subset [1, n'']$, the inverse image of the point x is $(S^1)^{\#I}$. The real manifold $D(r)^-$, together with the map $pr: D(r)^- \rightarrow D(r)$, will be called the real blow-up of $D(r)$ along V ; the inverse image $pr^{-1}(V)$ is called the set of directions of the real blow-up. Note

that the real blow-up $D(r)^- \xrightarrow{pr} D(r)$ is independent of the coordinates chosen in $D(r)$; this fact allows us to globalize our construction. Let $\{U_a\}$ be an open covering of M such that in each U_a with holomorphic coordinates $x_{a,i}, i \in [1, n]$, the subset $H \cap U_a$ may be given as the locus $\bigcup_{i=1}^{n''} \{x: x_{a,i} = 0\}$, and let $U_a^- \xrightarrow{pr} U_a$ be the real blow-up of U_a along $U_a \cap H$. We have then isomorphisms

$$pr_{ab}: pr_a^{-1}(U_a \cap U_b) \longrightarrow pr_b^{-1}(U_a \cap U_b)$$

and using them, we can patch together the local real blow-ups U_a^- to form a real analytic manifold $M^- = \bigcup_{pr_{ab}} U_a^-$ with a map $M^- \rightarrow M$. The manifold M^- , together with the map $pr: M^- \rightarrow M$, is called the real blow-up of M along H . By the construction, pr is an isomorphism away from $H \subset M$.

Let M^- be the real blow-up of M along H . For an open set U^- in M^- , we define $\mathcal{A}^-(U^-)$, $\mathcal{A}'^-(U^-)$ and $\mathcal{A}_0^-(U^-)$ as follows; if $pr(U^-) \cap H = \phi$,

$$\mathcal{A}^-(U^-) = \mathcal{A}'^-(U^-) = \mathcal{A}_0^-(U^-) = \mathcal{O}(pr(U^-)),$$

if $pr(U^-) \cap H \neq \phi$,

$$\mathcal{A}^-(U^-) = \mathcal{A}(pr(U^-) - H), \mathcal{A}'^-(U^-) = \mathcal{A}'(pr(U^-) - H),$$

$$\mathcal{A}_0^-(U^-) = \mathcal{A}_0(pr(U^-) - H).$$

Then, with the natural restriction mapping

$$i_{U^- \subset U'^-}: \mathcal{A}^-(U'^-) \longrightarrow \mathcal{A}^-(U^-)$$

for any open sets U^-, U'^- in M^- , $U^- \subset U'^-$, $\{\mathcal{A}^-(U^-), i_{U^- \subset U'^-}\}$ becomes a presheaf over M^- which satisfies the sheaf conditions. We denote by \mathcal{A}^- the associated sheaf on M^- and call \mathcal{A}^- the sheaf of germs of functions strongly asymptotically developable (as the variables tend to the normal crossing divisor H). In the same way we obtain the sheaf \mathcal{A}'^- and \mathcal{A}_0^- over M^- of germs of functions strongly asymptotically developable to $\mathcal{O}_{M \hat{\cap} H}$ and to 0, respectively. By the definition, there exists a natural inclusion i of \mathcal{A}_0^- to \mathcal{A}'^- and there exists a natural homomorphism FA of \mathcal{A}'^- to the inverse image $pr^*(\mathcal{O}_{M \hat{\cap} H})$ of $\mathcal{O}_{M \hat{\cap} H}$ which is obtained by taking the strongly asymptotic expansions.

Theorem 1. *The following short sequence of sheaves over M^- is exact:*

$$0 \longrightarrow \mathcal{A}_0^- \xrightarrow{i} \mathcal{A}'^- \xrightarrow{FA} pr^*(\mathcal{O}_{M \hat{\cap} H}) \longrightarrow 0.$$

From the exact sequence, we obtain the long exact sequences

$$0 \longrightarrow H^0(p^-, \mathcal{A}_0^-|_{p^-}) \longrightarrow H^0(p^-, \mathcal{A}'^-|_{p^-}) \longrightarrow H^0(p^-, pr^*(\mathcal{O}_{M \hat{\cap} H})|_{p^-}) \\ \longrightarrow H^1(p^-, \mathcal{A}_0^-|_{p^-}) \longrightarrow H^1(p^-, \mathcal{A}'^-|_{p^-}) \longrightarrow H^1(p^-, pr^*(\mathcal{O}_{M \hat{\cap} H})|_{p^-}) \cdots$$

for any point $p \in H$, where $p^- = pr^{-1}(p)$, and

$$0 \longrightarrow H^0(M^-, \mathcal{A}_0^-) \longrightarrow H^0(M^-, \mathcal{A}'^-) \longrightarrow H^0(M^-, pr^*(\mathcal{O}_{M \hat{\cap} H})) \\ \longrightarrow H^1(M^-, \mathcal{A}_0^-) \longrightarrow H^1(M^-, \mathcal{A}'^-) \longrightarrow H^1(M^-, pr^*(\mathcal{O}_{M \hat{\cap} H})) \cdots$$

Theorem 2. For any $p \in H$, the image of the mapping of $H^1(p^-, \mathcal{A}_0^-|_{p^-})$ to $H^1(p^-, \mathcal{A}'^-|_{p^-})$ is zero and so $H^1(p^-, \mathcal{A}_0^-|_{p^-})$ is isomorphic to $(\mathcal{O}_{M \hat{\cap} H})_p / (\mathcal{O}_p)$. Moreover, if $H^1(M, \mathcal{O}) = 0$ then the image of mapping of $H^1(M^-, \mathcal{A}_0^-)$ to $H^1(M^-, \mathcal{A}'^-)$ is zero and so $H^1(M^-, \mathcal{A}_0^-)$ is isomorphic to

$$H^0(M^-, pr^*(\mathcal{O}_{M \hat{\cap} H})) / H^0(M^-, \mathcal{A}'^-) = H^0(M, \mathcal{O}_{M \hat{\cap} H}) / H^0(M, \mathcal{O}).$$

Remark 1. We prove first that the mapping of $H^1(p^-, \mathcal{A}_0^-|_{p^-})$ to $H^1(p^-, \mathcal{A}'^-|_{p^-})$ is a zero mapping, from which we can deduce Theorem 2.

Put $\mathcal{A}'^-(*H) = \mathcal{A}'^- \otimes_{pr^*\mathcal{O}} pr^*(\mathcal{O}(*H))$, and $\mathcal{O}_{M \hat{\cap} H}(*H) = \mathcal{O}_{M \hat{\cap} H} \otimes_{\mathcal{O}} \mathcal{O}(*H)$. Then, the analogous results are valid for the following sequence of sheaves over M^- :

$$0 \longrightarrow \mathcal{A}_0^- \longrightarrow \mathcal{A}'^-(*H) \longrightarrow pr^*(\mathcal{O}_{M \hat{\cap} H}(*H)) \longrightarrow 0.$$

For $E = \mathcal{O}_{M \hat{\cap} H}, \mathcal{O}_{M \hat{\cap} H}(*H), \mathcal{A}'^-$ and $\mathcal{A}'^-(*H)$, denote by $GL(m, E)$ the sheaf of germs of invertible m -by- m matrices of the entries in E . Denote by I_m the m -by- m unit matrix and denote by $GL(m, \mathcal{A}^-)_{I_m}$ the sheaf over M^- of germs of m -by- m invertible matricial functions strongly asymptotically developable to I_m . Then, for the sequences

$$I_m \longrightarrow GL(m, \mathcal{A}^-)_{I_m} \longrightarrow GL(m, \mathcal{A}'^-) \longrightarrow pr^*(GL(m, \mathcal{O}_{M \hat{\cap} H})) \longrightarrow I_m, \\ I_m \longrightarrow GL(m, \mathcal{A}^-)_{I_m} \longrightarrow GL(m, \mathcal{A}'^-(*H)) \\ \longrightarrow pr^*(GL(m, \mathcal{O}_{M \hat{\cap} H}(*H))) \longrightarrow I_m,$$

the similar results are valid. These are extensions of the results due to Y. Sibuya [4], [5] and B. Malgrange [3] in one variable case. The detail and the applications are published elsewhere (see Majima [1], [2]).

References

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