41. *Null Strings and Null Solutions of Maxwell's Equations*

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1. The purpose of this note is to establish a relation between null strings and null solutions of Maxwell's equations (null electromagnetic fields). We will express Plücker's coordinates of the 2-dimensional surface swept out by a null string by means of a null electromagnetic field. The detailed proof will be given elesewhere.

Let (M, g) be Minkowski space with the affine coordinates x^μ, $\mu=0,1,2,3$. Throughout this note we adopt the summation convention. Namely, we suppress the summation sign every time that the summation has to be done on an index which appears twice in the term. To raise or lower indices, we use the formulas $X_\mu=g_{\mu\nu}X^\nu$ and $X^\mu=g^{\mu\nu}X_\nu$, where $g_{\mu\nu}$ are the components of the metric g and the matrix $(g^{\mu\nu})$ is the inverse of the matrix $(g_{\mu\nu})$.

A string is an extremal, i.e., a solution of the Euler-Lagrange equations, of the variational problem for 2-dimensional surfaces $\Sigma: x^\mu(\tau^1, \tau^2)$ in M:

$$\delta\left(\int L d\tau^1 d\tau^2\right)=0$$

with Lagrangean $L=v_{\mu\nu}v^{\mu\nu}$, where we define Plücker's coordinates $v^{\mu\nu}$ of the string Σ by

$$v^{\mu\nu}=(\partial x^\mu/\partial\tau^1)(\partial x^\nu/\partial\tau^2)-(\partial x^\nu/\partial\tau^1)(\partial x^\mu/\partial\tau^2).$$

A string on which $L=0$ is said to be null. In Minkowski space, Maxwell's equations take the Lorentz invariant form $dF=0=d{*F}$ for differential 2-forms F, where $*$ is Hodge operator. If a solution F of Maxwell's equations satisfies $g(F,F)=0=g(F,*F)$, we call it a null electromagnetic field.

Our main theorems are as follows.

Theorem 1. *Let a 2-dimensional surface $\Sigma: x^\mu(\tau^1, \tau^2)$ in Minkowski space be an analytic null string and let $x_{(0)}$ be a point on Σ. Then on some neighbourhood U_0 of $x_{(0)}$ in Minkowski space, a null electromagnetic field $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$ with a following property exists:*

(1) $\qquad v^{\mu\nu}(\tau^1, \tau^2)=F^{\mu\nu}(x(\tau^1, \tau^2)), \qquad x(\tau^1, \tau^2)=(x^\mu(\tau^1, \tau^2))\in U_0.$

Theorem 2. *Let $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$ be a null electromagnetic field. If a 2-dimensional surface $\Sigma: x^\mu(\tau^1, \tau^2)$ in Minkowski space satisfies* (1), *then Σ is a null string.*

Remark 1. There is a similar relation between spacelike strings, i.e., strings on which $L=-1$, and closed 2-forms F (see Rinke [3]). However, in this case $*F$ is not always closed.

Remark 2. We have shown in [1] that null electromagnetic fields can be constructed from holomorphic functions on an open set in the projective 3-space $P^3(C)$. Let F be a null electromagnetic field. Since $*F$ is closed and rank 2, there exist functions $T_1(x)$, $T_2(x)$ such that $*F=dT_1\wedge dT_2$. Let Σ be a maximal integral manifold of the system of Pfaffian equations $\{dT_1=0, dT_2=0\}$. Then we can take coordinates τ^1, τ^2 of Σ such that $\Sigma: x^\mu(\tau^1, \tau^2)$ satisfies (1).

2. Null strings and null electromagnetic fields. Here we recall some properties of null strings and null electromagnetic fields. On a null string there is a null tangent vector field σ^μ such that $v_{\mu\nu}\sigma^\nu=0$ $=*v_{\mu\nu}\sigma^\nu$ and every integral curve of σ^μ is a null straight line (see Schild [5]). Therefore a null string is generated by a 1-parameter family of null straight lines.

We can similarly characterize a null electromagnetic field. For any null electromagnetic field $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$, the conditions

$$(2)\qquad F_{\mu\nu}n^\nu=0, \qquad *F_{\mu\nu}n^\nu=0$$

determine a null vector field n^ν, which is geodesic and shear-free:

$$\{n_{[\mu;\nu]}-i*(n_{[\mu;\nu]})\}n^\nu=\{(1/2)n^{;\nu}_{;\nu}+\zeta\}n_\mu,$$
$$2n^{\mu;\nu}n_{(\mu;\nu)}=(n^{\nu}_{;\nu}+\zeta+\bar{\zeta})^2+(\zeta+\bar{\zeta})^2$$

with some complex function ζ, where square and round brackets denote skew symmetrization and symmetrization, respectively.

Conversely for any nonzero null vector field n^μ, equations (2) determine a null differential 2-form $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$ up to a change of amplitude and rotation, $F-i*F\rightarrow e^w(F-i*F)$ where w is an appropriate complex function. If the null vector field n^μ is geodesic and shear-free, then a null electromagnetic field subject to (2) exists. Let $F_{(0)}$ be one of them. Let l^μ, m^μ be the vector fields such that $g^{\mu\nu}=n^\mu l^\nu$ $+n^\nu l^\mu-\overline{m}^\mu m^\nu-\overline{m}^\nu m^\mu$, here l^μ is real while m^μ is complex and $\overline{m}^\mu$ is the complex conjugate of m^μ. Then we have:

Lemma 3. *Let $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$ be a differential 2-form which satisfies* (2). *Then F is a null electromagnetic field if and only if F has an expression as*

$$(3)\qquad F-i*F=e^w(F_{(0)}-i*F_{(0)})$$

where w is a solution of the equations

$$(4)\qquad w,_\mu n^\mu=0 \quad and \quad w,_\mu m^\mu=0$$

(see Robinson [4]).

3. Sketch of the proof of Theorem 1. Let a 2-dimensional surface $\Sigma: x^\mu(\tau^1, \tau^2)$ be an analytic null string. The null string Σ is generated by a 1-parameter family of null straight lines $\{L\}$. Using

Penrose's twistor theory [2], we have a one to one correspondence between 1-parameter analytic families of null straight line and holomorphic curves in $P^3(C)$. Let C be a holomorphic curve in $P^3(C)$ corresponding to $\{L\}$ and let S be holomorphic 2-dimensional surface in $P^3(C)$ such that $S \supset C$. Applying Kerr's theorem (see also Penrose [2]) to the surface S, we can define a null, geodesic and shear-free vector field n^μ which coincides with σ^μ on Σ. Take a null electromagnetic field $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$ such that $F_{\mu\nu} n^\nu = 0 = *F_{\mu\nu} n^\nu$. Then we have $v^{\mu\nu}(\tau^1, \tau^2) - i*v^{\mu\nu}(\tau^1, \tau^2) = e^{w(\tau^1,\tau^2)}(F^{\mu\nu}(x(\tau^1, \tau^2)) - i*F^{\mu\nu}(x(\tau^1, \tau^2)))$ with a function $w(\tau^1, \tau^2)$ on Σ. Substituting this into the Euler-Lagrange equations for the string motion, we see that $w(\tau^1, \tau^2)$ is constant along each null straight line which generates Σ. Therefore we can find a function $w(x^\mu)$ which satisfies (4) and coincides with $w(\tau^1, \tau^2)$ on Σ. The null electromagnetic field defined by (3) satisfies (1).

We can prove Theorem 2 by a direct calculation.

Remark 3. From any holomorphic surface $S \supset C$, we can construct a null electromagnetic field which satisfies (1).

References

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