5. *Singular Support of the Scattering Kernel for the Wave Equation Perturbed in a Bounded Domain*

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(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1983)

Introduction. Majda [4] obtained a representation of the scattering kernel $S(s, \theta, \omega)$ for the scattering by an obstacle $\mathcal{O}$ (in $\boldsymbol{R}^3$), and showed

$$(0.1)\quad \begin{array}{ll} \text{(i)} & \operatorname{supp} S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)], \\ \text{(ii)} & S(s, -\omega, \omega) \text{ is singular (not } C^\infty\text{) at } s=-2r(\omega), \end{array}$$

where $r(\omega)=\inf_{x \in \mathcal{O}} x\omega$. In the present note we shall consider the corresponding problems for the acoustic scattering by an inhomogeneous fluid.

Let $a_{ij}(x)=a_{ji}(x) \in C^\infty(\boldsymbol{R}^n)$ $(i, j=1, \cdots, n\ (n \geqq 2))$ satisfy

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqq \delta|\xi|^2, \qquad \xi \in \boldsymbol{R}^n,$$
$$a_{ii}(x)=1,\ a_{ij}(x)=0\ (i \neq j) \qquad \text{for } |x| \geqq r_0,$$

and set

$$Au=\sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j}u).$$

We consider the Cauchy problem

$$\begin{cases} (\partial_t^2-A)u(t, x)=0 & \text{in } \boldsymbol{R}^1 \times \boldsymbol{R}^n, \\ u(0, x)=f_1(x),\ \partial_t u(0, x)=f_2(x) & \text{on } \boldsymbol{R}^n. \end{cases}$$

In the same way as Lax and Phillips [1], [2], we define the scattering operator $S: L^2(\boldsymbol{R}^1 \times S^{n-1}) \to L^2(\boldsymbol{R}^1 \times S^{n-1})$ by $S=T_0^+(W^+)^{-1}W^-(T_0^-)^{-1}$, where T_0^+ (T_0^-) is the outgoing (incoming) translation representation associated with the unperturbed equation and $W^\pm$ are the wave operators (cf. Lax and Phillips [1], [2], the author [6]). S is represented with the distribution kernel $S(s, \theta, \omega)$ (called the scattering kernel) (cf. Majda [4], Lax and Phillips [3], the author [6]):

$$Sk(s, \theta)=\iint S(s-t, \theta, \omega)k(t, \omega)dtd\omega.$$

Let $v(t, x; \omega)$ $(\omega \in S^{n-1})$ be the solution of the equation

$$\begin{cases} (\partial_t^2-A)v=-2^{-1}(2\pi i)^{1-n}(\partial_t^2-A)\delta(t-x\omega) & \text{in } \boldsymbol{R}^1 \times \boldsymbol{R}^n, \\ v=0 & \text{for } t<-r_0. \end{cases}$$

$v(t, x; \omega)$ is a C^∞ function of x and ω with the value $\mathcal{S}'(\boldsymbol{R}_t^1)$.

Theorem 1. *Set*

$$S_0(s, \theta, \omega)=\int_{\boldsymbol{R}^n} (\partial_t^{n-2}\square v)(x\theta-s, x; \omega)dx \quad (\square=\partial_t^2-\Delta),$$

$$Kk=F^{-1}[(\operatorname{sgn}\sigma)^{n-1}(Fk)(\sigma)],$$

where F denotes the Fourier transformation in s. Then we have

$$Sk(s,\theta)=\iint S_0(s-t,\theta,\omega)k(t,\omega)dtd\omega+Kk(s,\theta).$$

Note that $S(s,\theta,\omega)=S_0(s,\theta,\omega)$ if $\theta\neq\omega$. In the scattering by an obstacle, the corresponding representation of the scattering kernel has been obtained (cf. Majda [4], the author [6]).

Using Theorem 1, we shall derive results corresponding to (0.1). Denote by $(q_\omega(t;y), p_\omega(t;y))$ the solution of the equation

$$\text{(0.2)}\qquad \begin{cases} \dfrac{dt}{dq}=-\partial_\xi\lambda_0^-(q,p), & \dfrac{dt}{dp}=\partial_x\lambda_0^-(q,p),\\ q|_{t=-r_0}=y\ \ (y\omega=-r_0), & p|_{t=-r_0}=\omega\ \ (\omega\in S^{n-1}), \end{cases}$$

where $\lambda_0^-(x,\xi)=-\left\{\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\right\}^{1/2}$.

Theorem 2. *For $\omega,\theta\in S^{n-1}$ set*

$$M_\omega(\theta)=\{y: y\omega=-r_0,\ \lim_{t\to\infty}p_\omega(t;y)=\theta\},$$
$$s_\omega(\theta)=\sup_{y\in M_\omega(\theta)}\lim_{t\to\infty}\{\langle q_\omega(t;y),\theta\rangle-t\}.$$

Then we have

$$\operatorname{sing\,supp} S_0(\cdot,\theta,\omega)\subset(-\infty,s_\omega(\theta)].$$

Theorem 3. *Let $n=2$. Then $S_0(s,\theta,\omega)$ is singular at $s=s_\omega(\theta)$.*

It is thought that $S_0(s,\theta,\omega)$ is singular at $s=s_\omega(\theta)$ also in the case of $n\geqq 3$. Our proof of Theorem 3, however, is not valid in this case. We note that in proof of Theorem 3 it does not suffice only to examine the wave front set of $v(t,x;\omega)$. We can prove Theorem 1 by the same procedure as in the author [6], whose idea is due to Majda [4], and so we omit its proof. We only give outlines of the proofs of Theorems 2 and 3.

§1. Proof of Theorem 2. Set $w(t,x)=v(t,x;\omega)+2^{-1}(2\pi i)^{1-n}\cdot\delta(t-x\omega)$. Then, by Theorem 1 we have

$$\text{(1.1)}\qquad S_0(s,\theta,\omega)=\int_{R^n}(\partial_t^{n-2}\Box w)(x\theta-s,x)dx.$$

Noting that $w(t,x)$ satisfies the equation

$$\text{(1.2)}\qquad \begin{cases} (\partial_t^2-A)w=0 & \text{in } \boldsymbol{R}^1\times\boldsymbol{R}^n,\\ w(-r_0,x)=2^{-1}(2\pi i)^{1-n}\delta(-r_0-x\omega) & \text{on } \boldsymbol{R}^n,\\ \partial_t w(-r_0,x)=2^{-1}(2\pi i)^{1-n}\delta'(-r_0-x\omega) & \text{on } \boldsymbol{R}^n, \end{cases}$$

by the well-known methods of the Fourier integral operators, we can know of the wave front set WF$[w(t,x)]$:

Proposition 1.1.

$$\mathrm{WF}[w(t,x)]=\{(t,x;\tau,\xi): t\in\boldsymbol{R}^1,\ x=q_\omega(t;y),\ \tau\in\boldsymbol{R}^1-\{0\},\ \xi=-\tau p_\omega(t;y)\}.$$

Since $\partial_t^2-A=\Box$ for $|x|\geqq r_0$, from this proposition we obtain

Lemma 1.2. *For any $\varepsilon>0$ there is a conic neighborhood Γ in*

$\boldsymbol{R}_\tau^1 \times \boldsymbol{R}_\xi^n - \{0\}$ *containing* $(1, -\theta)$ *and* $(-1, \theta)$ *such that if* $(t, x; \tau, \xi) \in \mathrm{WF}[w(t, x)] \cap \boldsymbol{R}_t^1 \times \boldsymbol{R}_x^n \times \Gamma$, (t, x) *satisfies*

$$|x| \leqq r_1 \quad \text{or} \quad x\theta - t \leqq s_\omega(\theta) + \varepsilon,$$

where r_1 *is some constant independent of* ε.

To prove Theorem 2, we have only to show that for any $\rho(s) \in C_0^\infty(\boldsymbol{R}^1)$ with $\mathrm{supp}[\rho] \subset (s_\omega(\theta), +\infty)$ $\bar{F}[\rho(s)S_0(s, \theta, \omega)](\sigma)$ decreases rapidly as $|\sigma| \to \infty$ $\left(\text{where } \bar{F}[k](\sigma) = \int e^{is\sigma} k(s) ds\right)$.

Lemma 1.3. *Let* $\alpha(x)$ *be a* C^∞ *function on* $\boldsymbol{R}^n$ *such that* $\alpha(x) = 1$ *for* $|x| \geqq \tilde{r}$ ($\tilde{r}$ *is any constant*). *Then, for any* $\rho(s) \in C_0^\infty(\boldsymbol{R}^1)$ *we have*

$$\bar{F}[\rho(s)S_0(s, \theta, \omega)](\sigma) = \mathcal{F}[\rho(x\theta - t)\partial_t^{n-2} \Box(\alpha w)](\sigma, -\sigma\theta),$$

where $\mathcal{F}$ *denotes the Fourier transformation in* (t, x).

Take the function $\alpha(x)$ in Lemma 1.3 so that $\alpha(x) = 0$ for $|x| \leqq r_1$ and $\alpha(x) = 1$ for $|x| \geqq r_1 + 1$ (r_1 is the constant in Lemma 1.2). It follows from Lemma 1.3 that

$$\bar{F}[\rho(s)S_0(s, \theta, \omega)](\sigma) = \mathcal{F}[\rho(x\theta - t)\chi(D_t, D_x)\partial_t^{n-2} \Box(\alpha w)](\sigma, -\sigma\theta) + 0(|\sigma|^{-\infty}),$$

where $\chi(\tau, \xi)$ is a C^∞ function homogeneous of order 0 satisfying $\mathrm{supp}[\chi] \subset \Gamma$ (Γ is the set in Lemma 1.2) and $\chi(\tau, \xi) = 1$ in a neighborhood of $(1, -\theta)$, $(-1, \theta)$. Lemma 1.2 implies that

$$\mathrm{WF}[\rho(x\theta - t)\chi(D_t, D_x)\partial_t^{n-2} \Box(\alpha w)] = \phi.$$

Therefore Theorem 2 is obtained.

§ 2. Proof of Theorem 3. It suffices to show that for any small $\varepsilon (> 0)$ there exist some C^∞ function $\rho(s)$ and a real number m such that $\mathrm{supp}[\rho] \subset [s_\omega(\theta) - \varepsilon, \ s_\omega(\theta) + \varepsilon]$ and $(1 + |\sigma|)^m \bar{F}[\rho(s)S_0(s, \theta, \omega)](\sigma) \notin L^2(\boldsymbol{R}^1)$. We cannot justify this assertion only by examining $\mathrm{WF}[w]$ (w is the solution of (1.2)).

Let us consider only the case that $M_\omega(\theta)$ is bounded. Denote by $\tilde{w}(t, x)$ the solution of (1.2) with the different initial data $\tilde{w}(-r_0, x) = \gamma(x) w(-r_0, x)$, $\partial_t \tilde{w}(-r_0, x) = \gamma(x) \partial_t w(-r_0, x)$ on $\boldsymbol{R}^n$, where $\gamma(x)$ is a C^∞ function such that $\mathrm{supp}[\gamma]$ is contained in a sufficiently small neighborhood of $\mathrm{M}_\omega(\theta)$ and that $\gamma(x) = 1$ on a neighborhood of $M_\omega(\theta)$. Let $\alpha(x)$ be the function in the proof of Theorem 2. Then, if $\mathrm{supp}[\rho]$ is small enough, by Lemma 1.3 we have

$$\bar{F}[\rho(s)S_0(s, \theta, \omega)](\sigma) = \mathcal{F}[\rho(x\theta - t)\partial_t^{n-2} \Box(\alpha\tilde{w})](\sigma, -\sigma\theta) + 0(|\sigma|^{-\infty}).$$

Furthermore it is seen that if $\tilde{t}$ is large enough we obtain for any integer $N (\geqq 0)$

$$\begin{aligned} &\mathcal{F}[\rho(x\theta - t)\partial_t^{n-2} \Box(\alpha\tilde{w})](\sigma, -\sigma\theta) \\ &\quad = \mathcal{F}'\left[\sum_{j=0}^{N} \alpha_j(x) \sigma^{n-1-j} \tilde{w}(\tilde{t}, x)\right](-\sigma\theta) + 0(|\sigma|^{-N+l}). \end{aligned}$$

Here, $\mathcal{F}'$ denotes the Fourier transformation in x, l is an integer independent of N and $\mathrm{supp}[\alpha_j] \subset \{x : r_1 \leqq |x| \leqq r_1 + 1\}$ (r_1 is the constant in Lemma 1.2). Let $(q(t; s, x, \xi), p(t; s, x, \xi))$ be the solution of (0.2) with the different initial data $q|_{t=s} = x$, $p|_{t=s} = \xi$.

Lemma 2.1. *Let s and t be arbitrary constants in $[-\gamma_0, \bar{t}]$ satisfying $|s-t|\leqq\delta$. Assume that $\varphi(x)$ is any real-valued C^∞ function with $\varphi(q(t; s, y, \eta))=0$, $\partial_x\varphi(q(t; s, y, \eta))=0$ and that $\beta(x)$ be any C^∞ function with* $\mathrm{supp}[\beta]\subset\{x: |x-q(t; s, y, \eta)|<\varepsilon\}$. *Then, if δ and ε are small enough, there is a real-valued C^∞ function $\psi(x)$ such that $\psi(y)=0$, $\partial_x\psi(y)=0$ and that for any integer N $(\geqq 0)$*

$$\begin{aligned}\mathcal{F}'[e^{i\sigma\varphi(x)}\beta(x)\tilde{w}(t, x)](\sigma p(t; s, y, \eta)) &= \exp\{i\sigma y\eta - i\sigma p(t; s, y, \eta)q(t; s, y, \eta)\} \\ &\quad\times\mathcal{F}'\left[e^{i\sigma\psi(x)}\sum_{j=0}^{N}\chi_j(x)\sigma^{-j}\tilde{w}(s, x)\right](\sigma\eta)+0(|\sigma|^{-N+l}),\end{aligned}$$

where l is an integer independent of N and $\chi_j(x)$ is a C^∞ function such that $\lim_{\varepsilon\to+0}\mathrm{dis}(y, \mathrm{supp}[\chi_j])=0$.

Take a sufficiently fine partition of unity $\{\beta_k(x)\}$ on $\boldsymbol{R}^n$, and apply Lemma 2.1 to each $\mathcal{F}'[\alpha_j(x)\beta_k(x)\tilde{w}(\bar{t}, x)](-\sigma\theta)$ repeatedly (divide $[-\gamma_0, \bar{t}]$ into many fine intervals and use Lemma 2.1 on each interval). Then it is seen that there are C^∞ functions $\{\psi_k(x)\}$ and $\{\chi_{kj}(x)\}$ such that

$$\begin{aligned}\mathcal{F}[\rho(x\theta-t)\partial_t^{n-2}\Box(\alpha\tilde{w})](\sigma, -\sigma\theta) &= \exp\{-i\sigma(r_0+\bar{t}+s_\omega(\theta))\}\sigma^{n-1} \\ &\quad\times\sum_{k=1}^{N'}\mathcal{F}'\left[e^{i\sigma\psi_k(x)}\left(\sum_{j=0}^{N}\chi_{kj}(x)\sigma^{-j}\right)\tilde{w}(-r_0, x)\right](-\sigma\omega)+0(|\sigma|^{-N+l}).\end{aligned}$$

We see that if we choose α and ρ appropriately the above ψ_k and χ_{kj} satisfy all the assumptions of the following lemma, and therefore Theorem 3 is obtained.

Lemma 2.2. *Assume that $\{\psi_k(x)\}_{k=1,\cdots,N'}$ are real-valued C^∞ functions on $\boldsymbol{R}^2$ such that $\psi_k(y_k)=0$, $\partial_x\psi_k(y_k)=0$ $(y_k\omega=-r_0)$. Let $\gamma(x)$ and $\{\chi_{kj}(x)\}_{k=1,\cdots,N';\ j=0,\cdots,N}$ belong to $C_0^\infty(\boldsymbol{R}^2)$ and satisfy $\gamma(x)\geqq 0$, $\sum_{k=1}^{N'}\chi_{k0}(x)\geqq 0$ and $\sum_{k=1}^{N'}\chi_{k0}(y_k)\gamma(y_k)>0$. Then for some m $(<1/2)$ we have*

$$(1+|\sigma|)^m\sum_{k=1}^{N'}\mathcal{F}'\left[e^{i\sigma\psi_k(x)}\left(\sum_{j=0}^{N}\chi_{kj}(x)\sigma^{-j}\right)\gamma(x)\delta(-r_0-x\omega)\right](-\sigma\omega)\notin L^2(\boldsymbol{R}_\sigma^1).$$

This lemma is not correct in the case of $n\geqq 3$. Its proof is similar to that of Theorem 1 in the author [5].

References

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