35. Singular Cauchy Problems for Second Order Partial Differential Operators with Non-Involutory Characteristics

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We denote by (x, y) the variables of C^{n+1}, where $x \in C$ and $y=(y_1, y') \in C\times C^{n-1}$, and by (ξ, η) the dual variables of (x, y). We consider partial differential operators written in the following form:

$$P(x, y, \partial/\partial x, \partial/\partial y)=\sum_{i+|\alpha|\leqslant 2} x^{\kappa(i,\alpha)}a_{i\alpha}(x, y)(\partial/\partial x)^i(\partial/\partial y)^\alpha.$$

Here $\kappa(i, \alpha)$, $i+|\alpha|\leqslant 2$, are integers defined by

$$\kappa(i, \alpha)=\begin{cases} q|\alpha| & i+|\alpha|=2 \\ q' & i=0, |\alpha|=1 \\ 0 & \text{otherwise,} \end{cases}$$

where q and q' are integers which satisfy $0\leqslant q'\leqslant q-2$. Furthermore, $a_{i\alpha}(x, y)$, $i+|\alpha|\leqslant 2$, are holomorphic at the origin, and $a_{2,0}=1$.

Remark. If $q'=q-1$, the above operators are said to satisfy Levi condition. Several authors considered singular Cauchy problems for such operators (see Nakane [1], Takasaki [2], and Urabe [4]). Perhaps we can also treat this case, but this requires some modifications which are not trivial. Thus we consider only the case of $q'\leqslant q-2$.

We assume that the equation

$$\sum_{i+|\alpha|=2} x^{q|\alpha|}a_{i\alpha}(x, y)\xi^i\eta^\alpha=0$$

has two roots $\xi=x^q\lambda_i(x, y, \eta)$, $i=1, 2$, where $\lambda_i(x, y, \eta)$, $i=1, 2$, are holomorphic at $x=0$, $y=0$, $\eta=(1, 0, \cdots, 0)$ and homogeneous of degree 1 in η. Furthermore we assume that

$$\lambda_1(x, y, \eta)\neq\lambda_2(x, y, \eta)$$

at $x=0$, $y=0$, $\eta=(1, 0, \cdots, 0)$.

Our purpose is to solve the following singular Cauchy problems:

$$(1)\qquad \begin{cases} Pu(x, y)=0 \\ (\partial/\partial x)^i u(0, y)=\mathring{u}_i(y) \qquad i=0, 1. \end{cases}$$

Here $\mathring{u}_i(y)$, $i=0, 1$, are multivalued holomorphic functions defined on $\{y \in C^n; |y_j|<R, j=1, 2, \cdots, n, y_1\neq 0\}$ with some $R>0$, and satisfy

$$|\mathring{u}_i(y)|\leqslant C\exp\{C|y_1|^{-(q-1-q')/(q+1)}\}$$

with some $C>0$ there.

Let us define $\varphi_i(x, y)$, $i=1, 2$, by

$$\begin{cases}(\partial/\partial x)\varphi_i(x, y)-x^q\lambda_i(x, y, \nabla_y\varphi_i(x, y))=0\\ \varphi_i(0, y)=y_1,\end{cases}$$

and $\psi_i(x, y')$, $i=1, 2$, by

$$\varphi_i(x, y)=0 \qquad \text{if and only if } y_1=\psi_i(x, y').$$

Then we have the following

Theorem. *Let $r>0$ be small enough, and $\theta\in\boldsymbol{R}$ be arbitrary. We define $\omega_{r,\theta}=\omega'_{r,\theta}\cup\omega''_{r,\theta}$ by*

$$\omega'_{r,\theta}=\Big\{(x, y)\in\boldsymbol{C}\times\boldsymbol{C}^n;\ |x|<r,\ |y_j|<r,\ j=1, 2, \cdots, n,$$
$$|\arg(y_1-\psi_i(x, y'))-\theta|<\frac{\pi}{2}+r,\ i=1, 2\Big\}$$
$$\omega''_{r,\theta}=\Big\{(x, y)\in\boldsymbol{C}\times\boldsymbol{C}^n;\ |x|<r,\ |y_j|<r,\ j=1, 2, \cdots, n,$$
$$|\arg(y_1-\psi_i(x, y'))-\theta-\pi|<\frac{\pi}{2}+r,\ i=1, 2\Big\}.$$

Then there exists a unique solution $u(x, y)$ of (1) which satisfies

$$|u(x, y)|\leqslant C\sum_{i=1,2}\exp\{C|\varphi_i(x, y)|^{-(q-1-q')/(q+1)}\}$$

with some $C>0$ on $\omega_{r,\theta}$.

Remark. Let us fix $(x, y')\in\boldsymbol{C}\times\boldsymbol{C}^{n-1}$ arbitrarily. Let us define θ by $\theta=\arg(\psi_1(x, y')-\psi_2(x, y'))+\pi/2$. Then it is easy to see that $\omega_{r,\theta}$ is a domain in the universal covering space of $\omega_r=\{(x, y)\in\boldsymbol{C}\times\boldsymbol{C}^n;\ |x|<r, |y_j|<r, j=1, 2, \cdots, n, \varphi_i(x, y)\neq 0, i=1, 2\}$ which projects to the whole base space ω_r. However, we cannot construct the solution on an arbitrary domain in the universal covering space of ω_r.

Remark. Furthermore, we can give a concrete representation of the solution. Let $\theta\in\boldsymbol{R}$ and $l\in\boldsymbol{Z}$ be arbitrary. We define θ_0 by

$$\theta_0=-\arg\{[\lambda_2(x, y, \eta)-\lambda_1(x, y, \eta)]_{x=0,\, y=0,\, \eta=(1,0,\cdots,0)}\}.$$

We define $V^i_{r,\theta,l}$, $i=1, 2$, by

$$V^i_{r,\theta,l}=\Big\{(x, y)\in\boldsymbol{C}\times\boldsymbol{C}^n;\ |x|<r,\ |y_j|<r,\ j=1, 2, \cdots, n,$$
$$\Big|(q+1)\arg x-(\theta_0+\pi l+\theta)-\frac{\pi}{2}\Big|<\frac{3}{4}\pi,$$
$$|\arg(y_1-\psi_i(x, y'))-\theta|<\frac{\pi}{2}+r\Big\}.$$

Then there exist holomorphic functions $v^i_{\theta,l}(x, y)$, $i=1, 2$, defined on $V^i_{r,\theta,l}$ which satisfy

$$|v^i_{\theta,l}(x, y)|\leqslant C\exp\{C|y_1-\psi_i(x, y')|^{-(q-1-q')/(q+1)}\}$$

with some $C>0$ on $V^i_{r,\theta,l}$ and

$$(2)\qquad v^1_{\theta,l}(x, y)+v^2_{\theta,l}(x, y)=u(x, y)$$

on $V^1_{r,\theta,l}\cap V^2_{r,\theta,l}$. Since $\bigcup_{l\in\boldsymbol{Z}}(V^1_{r,\theta,l}\cap V^2_{r,\theta,l})=\omega'_{r,\theta}$, the solution $u(x, y)$ is decomposed into the form (2) on $\omega'_{r,\theta}$, depending on $\arg x$. We can give a concrete representation of $v^i_{r,\theta,l}(x, y)$ using some class of operators. Analogous results hold also on $w''_{r,\theta}$. The details will be given in Uchikoshi [3].

References

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