31. Short Term Consistency Relations for Doubly Polynomial Splines

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By making use of the B-spline $Q_{m+1}(x)$:

$$Q_{m+1}(x)=(1/m!)\sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}(x-i)_+^m$$

where

$$(x-i)_+^m=\begin{cases}(x-i)^m & \text{for } x\geq i\\ 0 & \text{for } x<i,\end{cases}$$

we consider a quartic spline $s(x)$ of the form:

$$s(x)=\sum_{i=-4}^{n-1}\alpha_i Q_5(x/h-i),\qquad nh=1.$$

Then the following short term consistency relation has been obtained by Usmani ([6]):

$$(*)\qquad (s_{i+1}-2s_i+s_{i-1})=(h^2/12)(s''_{i+1}+10s''_i+s''_{i-1})$$

where $s_i=s(ih)$ and $s''_i=s''(ih)$. The above relation has been generalized for even degree polynomial splines ([3]). For odd degree polynomial splines, we also have short term consistency relations at mid-points ([4]). For example, let $s(x)$ be a cubic, then

$$(**)\qquad (s_{i+3/2}-2s_{i+1/2}+s_{i-1/2})=(h^2/24)(s''_{i+3/2}+22s''_{i+1/2}+s''_{i-1/2})$$

where $s_{i+1/2}=s((i+1/2)h)$ and $s''_{i+1/2}=s''((i+1/2)h)$.

In the present paper we shall generalize the above relations (*) and (**) for doubly polynomial splines.

Let $s(x,y)$ be a polynomial spline of the form:

$$s(x,y)=\sum_{i,j=-m}^{n-1}\alpha_{i,j}Q_{m+1}(x/h-i)Q_{m+1}(y/h-j).$$

Then we have

Theorem 1. *If m is even and $k,l\ (\leq m-2)$ are also even, we have*

$$\sum_{i,j=0}^{m-2}c_{i,j}^{(k,l)}s_{i,j}=h^{k+l}\sum_{i,j=0}^{m-2}c_{i,j}^{(0,0)}s_{i,j}^{(k,l)}$$

where

$$s_{i,j}^{(k,l)}=\frac{\partial^{k+l}}{\partial x^k\partial y^l}s(ih,jh)$$

$$c_{i,j}^{(k,l)}=\{Q_{m+1}^{(k)}(m-i)-Q_{m+1}^{(k)}(m-i+1)+\cdots\}\times\{Q_{m+1}^{(l)}(m-j)-Q_{m+1}^{(l)}(m-j+1)+\cdots\}.$$

Proof. The following m^2-term consistency relation holds:

$$\text{(E)}\qquad \sum_{i,j=0}^{m-1} Q_{m+1}^{(k)}(m-i)Q_{m+1}^{(l)}(m-j)s_{p+i,r+j}$$
$$=h^{k+l}\sum_{i,j=0}^{m-1} Q_{m+1}(m-i)Q_{m+1}(m-j)s_{p+i,r+j}^{(k,l)} \quad ([1]).$$

Since

$$Q_{m+1}(x)\equiv 0 \qquad \text{for } x\le 0,\ x\ge m+1$$
$$Q_{m+1}(x)\equiv Q_{m+1}(m+1-x),$$

for $i\ge m-1$;

$$c_{i,j}^{(k,l)}=(-1)^{i-m+1}\{Q_{m+1}^{(k)}(1)-Q_{m+1}^{(k)}(2)+\cdots-Q_{m+1}^{(k)}(m)\}$$
$$\times\{Q_{m+1}^{(l)}(m-j)-Q_{m+1}^{(l)}(m-j+1)+\cdots\}$$
$$=0 \qquad \text{for even } k,$$

for $j\ge m-1$;

$$c_{i,j}^{(k,l)}=0 \qquad \text{for even } l.$$

Hence, *an alternating sum* obtained by

(i) writing down equation (E) with $(p,r)=(0,0)$, substracting equation (E) with $(p,r)=(1,0)$, adding equation (E) with $(p,r)=(2,0)$ and so on,

(ii) substracting equation (E) with $(p,r)=(0,1)$, adding equation (E) with $(p,r)=(1,1)$ and so on,

(iii) continuating these processes,

is equal to the short term consistency relation.

As an example of the above relation, let s be a doubly quartic spline, then

$$(1/24)\{s_{i+1,j+1}+s_{i+1,j-1}+s_{i-1,j+1}+s_{i-1,j-1}$$
$$+4(s_{i+1,j}+s_{i,j+1}+s_{i,j-1}+s_{i-1,j})-20s_{i,j}\}$$
$$=(h/24)^2\{\Delta s_{i+1,j+1}+\Delta s_{i+1,j-1}+\Delta s_{i-1,j+1}+\Delta s_{i-1,j-1}$$
$$+10(\Delta s_{i+1,j}+\Delta s_{i,j+1}+\Delta s_{i,j-1}+\Delta s_{i-1,j})+100\Delta s_{i,j}\}.$$

This relation is useful for the numerical solution of a boundary value problem $\Delta u=f$ and the discretization error of this nine-point difference scheme is $O(h^6)$ ([2]).

If m is odd, we have the following

Theorem 2. *If m is odd and k,l $(\le m-1)$ are even, we have the short term consistency relation at mid-points:*

$$\sum_{i,j=0}^{m-1} d_{i,j}^{(k,l)}s_{i+1/2,j+1/2}=h^{k+l}\sum_{i,j=0}^{m-1} d_{i,j}^{(0,0)}s_{i+1/2,j+1/2}^{(k,l)}$$

where

$$s_{i+1/2,j+1/2}^{(k,l)}=\frac{\partial^{k+l}}{\partial x^k\partial y^l}s((i+1/2)h,(j+1/2)h)$$
$$d_{i,j}^{(k,l)}=\{Q_{m+1}^{(k)}(m+1/2-i)-Q_{m+1}^{(k)}(m+3/2-i)+\cdots\}$$
$$\times\{Q_{m+1}^{(l)}(m+1/2-j)-Q_{m+1}^{(l)}(m+3/2-j)+\cdots\}.$$

Let s be a doubly cubic spline. Then from above we have

$$(1/48)\{s_{i+3/2,j+3/2}+s_{i+3/2,j-1/2}+s_{i-1/2,j+3/2}+s_{i-1/2,j-1/2}$$
$$+10(s_{i+3/2,j+1/2}+s_{i+1/2,j+3/2}+s_{i+1/2,j-1/2}+s_{i-1/2,j+1/2})-44s_{i+1/2,j+1/2}\}$$

$$=(h/48)^2\{\Delta s_{i+3/2,j+3/2}+\Delta s_{i+3/2,j-1/2}+\Delta s_{i-1/2,j+3/2}+\Delta s_{i-1/2,j-1/2}$$
$$+22(\Delta s_{i+3/2,j+1/2}+\Delta s_{i+1/2,j+3/2}+\Delta s_{i+1/2,j-1/2}+\Delta s_{i-1/2,j+1/2})$$
$$+484\Delta s_{i+1/2,j+1/2}\}.$$

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