30. On 4-Manifolds Fibered by Tori. II

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This is a sequel to our previous note [2]. We will prove a signature formula for torus fibrations (Theorem 6) by combining Novikov additivity and W. Meyer's theorem [4]. This formula seems useful, especially in the study of singular fibers. Some computations will be presented. Also we will give a necessary condition for the existence of good torus fibrations in the sense of [3].

§ 5. **The signature formula.** Throughout the note, all manifolds will be compact, oriented and smooth. Sign (M) will denote the signature of the homological intersection form $H_2(M;Z)\times H_2(M;Z)\to Z$, where M is a connected 4-manifold with or without boundary.

Let F_i be a singular fiber of a torus fibration $f_i: M_i^4\to B_i^2$, for each $i=1,2$. Let $\{p_i\}=f_i(F_i)$.

Definition. F_1 and F_2 are said to be *topologically equivalent* if there exist neighborhoods U_1, U_2 of p_1, p_2 in Int (B_1^2), Int (B_2^2), respectively, and orientation preserving homeomorphisms $h: U_1\to U_2$ and $H: f_1^{-1}(U_1)\to f_2^{-1}(U_2)$, so that (i) $h(p_1)=p_2$ and (ii) $h\circ f_1=f_2\circ H$.

Let $\mathcal{S}$ denote the totality of topological equivalence classes of singular fibers. Let $(1/3)Z=\{m/3\mid m\in Z\}\subset \boldsymbol{Q}$.

Theorem 6. *There exists a (practically computable) function* σ: $\mathcal{S}\to(1/3)Z$ *with the following property: If* $\{F_1,\cdots,F_r\}$ *is the set of all the singular fibers of a given torus fibration* $f: M^4\to B^2$ *with* M^4 *closed, then* Sign $(M)=\sum_{i=1}^r \sigma(F_i)$ *holds.*

Consider a situation in which a singular fiber F_0 splits into several singular fibers $F_1',\cdots,F_r'$ through a certain deformation process (cf. [2], § 3). In that case, we have the following:

Corollary 6.1. $\sigma(F_0)=\sum_{i=1}^r \sigma(F_i')$.

Remark. As we see above, each singular fiber behaves as if it has 'fractional signature'.

Proof of Theorem 6. Let $\omega: E\to X$ be any torus bundle over a connected surface X. Let $\partial X=C_1\cup\cdots\cup C_r$. Let $\alpha_i\in$ SL $(2,Z)$ be a monodromy matrix of the restriction $E\mid C_i$ of E to C_i, where C_i is oriented so that the orientation is concordant with that of X. The conjugacy class of α_i is uniquely determined.

Let ϕ: SL $(2,Z)\to(1/3)Z$ be Meyer's class function whose explicit formula (containing the Dedekind sum) is found in [4], § 5.

Theorem 7 (Meyer [4], Satz 5). $\mathrm{Sign}\,(E)=\sum_{i=1}^{r}\phi(\alpha_i)$.

Remark. $\tau(E, \partial E)$ in [4], Satz 5, is equal to $-\mathrm{Sign}\,(E)$. See the equation (6) in [4].

Now let $f: M\to B$ be a torus fibration, M being closed. Let $\{F_1, \cdots, F_r\}$ be the set of all the singular fibers of f, D_i a small disk in B centred at $p_i=f(F_i)$, $i=1, \cdots, r$. We remove the saturated neighborhoods $N_i=f^{-1}(D_i)$, $i=1, \cdots, r$, from M. Then we get a torus bundle $f|E: E\to X$, where $E=M-\bigcup_{i=1}^{r}\mathrm{Int}\,(N_i)$, $X=B-\bigcup_{i=1}^{r}\mathrm{Int}\,(D_i)$. Let β_i be a monodromy matrix of a singular fiber F_i. Here we notice our convention that when we speak of a monodromy matrix of a singular fiber F_i, it is always computed w. r. t. the orientation of ∂D_i which is concordant with that of D_i (see [3], § 4). Therefore β_i is conjugate to the inverse of α_i defined above, and we have $\phi(\beta_i)=-\phi(\alpha_i)$ (see the formula (42) in [4]).

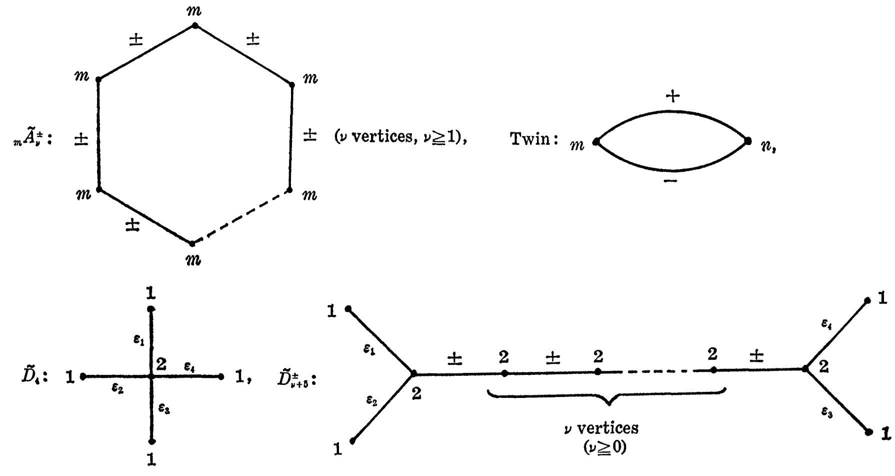
By Novikov additivity and Theorem 7, it follows that

$$\mathrm{Sign}\,(M)=\mathrm{Sign}\,(E)+\sum_{i=1}^{r}\mathrm{Sign}\,(N_i)$$
$$=\sum_{i=1}^{r}\{-\phi(\beta_i)+\mathrm{Sign}\,(N_i)\}.$$

Defining $\sigma(F_i)$ to be $-\phi(\beta_i)+\mathrm{Sign}\,(N_i)$, we have the desired formula.

§ 6. Computations in good torus fibrations. A good torus fibration (GTF) is a torus fibration whose singular fibers are of normal type in the sense of [2], § 4. Such singular fibers without removable linear branches are classified in [2], § 4, [3], Thm. 3.1, and their monodromy matrices are known, [3], Thm. 4.1. Thus their σ-numbers can be computed.

Before stating the results of computation we introduce some refined notation for the classes of singular fibers,



$\tilde{E}_6^{\pm}$: $\tilde{E}_6$ with $\varepsilon_3 m_3 \equiv \mp 1$ (mod 3),
$\tilde{E}_7^{\pm}$: $\tilde{E}_7$ with $\varepsilon_3 m_3 \equiv \mp 1$ (mod 4),
$\tilde{E}_8^{\pm}$: $\tilde{E}_8$ with $\varepsilon_3 m_3 \equiv \mp 1$ (mod 6),

where in the last three classes, ε_3(or m_3) denotes the sign of the edge (or the multiplicity of the vertex) in the (p_3)-branch which is adjacent to the m-vertex. Here we are referring to the graph on p. 300 of [2]. (Cf. [3], Thm. 4.1.)

Theorem 8. *We have the following table* (in which $\Sigma\varepsilon$ stands for the sum of the signs of all the edges):

class of F		$\sigma(F)$	euler number $\chi(F)$
${}_mI_0$		0	0
$\tilde{A}$	${}_m\tilde{A}_\nu^{\pm}$	$\mp(2/3)\nu$	ν
	Twin	0	2
$\tilde{D}$	$\tilde{D}_4$	$-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$	6
	$\tilde{D}_{\nu+5}^{\pm}$	$\mp(2/3)(\nu+1)-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$	$\nu+7$
$\tilde{E}_6^{\pm}$		$\pm(2/3)-\Sigma\varepsilon$	(number of vertices)+1
$\tilde{E}_7^{\pm}$		$\pm 1-\Sigma\varepsilon$	
$\tilde{E}_8^{\pm}$		$\pm(4/3)-\Sigma\varepsilon$	

Although they are not necessarily of normal type, we can make similar computations for the singular fibers of complex elliptic surfaces, either by using the Table I in [1], p. 604 or by blowing down exceptional curves in the singular fibers of normal type:

Corollary 8.1 (for the symbols ${}_m\mathrm{I}_b$, ${}_m\mathrm{I}_b^*$, II, etc., see [1]).

F	${}_m\mathrm{I}_0$	${}_m\mathrm{I}_b$	${}_m\mathrm{I}_b^*$	II	II*	III	III*	IV	IV*
$\sigma(F)$	0	$-(2/3)b$	$-(2/3)(b+6)$	$-4/3$	$-20/3$	-2	-6	$-8/3$	$-16/3$

Comparing the table with the values of euler numbers, we obtain $\sigma(F)=-(2/3)\chi(F)$. This is considered as a 'local form' of a known relation for each elliptic surface M which contains no exceptional curves in its fiber: Sign $(M)=-(2/3)\chi(M)$.

Returning to our torus fibration, we have the following theorem by a closer analysis of singular fibers:

Theorem 9. *Let $F=\Sigma m_i\Theta_i$ be a singular fiber of normal type. If the self-intersection number $\Theta_i\cdot\Theta_i$ is an even integer for each Θ_i, then $|\sigma(F)|\leqq(2/3)\chi(F)$.*

Corollary 9.1. *Let $f: M\to B$ be a GTF. Suppose that M is closed and $w_2(M)=0$, then* |Sign $(M)|\leqq(2/3)\chi(M)$.

For example, a connected sum of $K3$ surfaces $M'=K3\#K3$ does not admit any GTF, because $w_2(M')=0$, Sign $(M')=-32$ and $\chi(M')=46$.

On the other hand M' has a general torus fibration [2], Thm. 4. Thus the torus fibration cannot be deformed into GTF. This means that the 'program' stated in [3], § 1 must be amended slightly.

References

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[4] W. Meyer: Die Signatur von Flächenbündeln. Math. Ann., **201**, 239–264 (1973).