16. The Exponential Calculus of Microdifferential Operators of Infinite Order. I

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1. Introduction. Let X be an open set in C^n and T^*X be its cotangent bundle. Let $x=(x_1, \cdots, x_n)$ be a coordinate system of X. Then a point in T^*X is denoted by $(x, \langle \xi, dx \rangle)=(x, \xi)=(x_1, \cdots, x_n; \xi_1, \cdots, \xi_n)$. We denote by $\mathcal{E}^R$ the sheaf on T^*X of holomorphic microlocal operators (cf. [2], [4], [5], [7]). The sheaf $\mathcal{E}^R$ includes the sheaf $\mathcal{E}^\infty$ of microdifferential operators (=analytic pseudodifferential operators) of infinite order. Let $\mathring{x}^*=(\mathring{x}, \mathring{\xi})$ be a point in T^*X. A holomorphic microlocal operator F in $\mathcal{E}^R_{\mathring{x}^*}$ is determined by a holomorphic function $F(x, \xi)$ which is defined in some conic neighborhood Ω of $\mathring{x}^*$ and which satisfies the following condition: for each $\varepsilon>0$ and each compactly generated cone $\Omega' \subset \Omega$ there exists a positive constant C such that $|F(x, \xi)| \le C \exp(\varepsilon |\xi|)$ for $(x, \xi) \in \Omega'$. Note that we often neglect a bounded subset of a conic set in T^*X. The holomorphic function $F(X, \xi)$ is said to be the symbol of the operator F (cf. [2], [6]). Then we write $F= :F(x, \xi):$ or $F=F(x, D_x)$, where $D_x=(D_1, \cdots, D_n)$, $D_j=\partial/\partial x_j$.

The aim of this note is to give the exponential rules of symbols of holomorphic microlocal operators under suitable growth conditions. That is, we give explicit formulae for leading terms of $r(x, \xi)$ which satisfies

$$:\exp\{p(x, \xi)\}: \, :\exp\{q(x, \xi)\}:=\, :\exp\{r(x, \xi)\}:. \tag{1.1}$$

Here $p(x, \xi)$, $q(x, \xi)$ are given symbols which satisfy appropriate conditions and the left-hand side is the composite operator of $:\exp\{p(x, \xi)\}:$ and $:\exp\{q(x, \xi)\}:$.

Such formulae enable us to extend the result of our previous article [2] on invertibility of operators.

2. Statement of results. Let $F= :F(x, \xi):$ be a holomorphic microlocal operator with symbol $F(x, \xi)$ defined in $\Omega \ni \mathring{x}^*$. Let ρ be a real number such that $0 \le \rho < 1$. The operator F, or the symbol $F(x, \xi)$, is said to be of growth order at most (ρ) if for each compactly generated cone $\Omega' \subset \Omega$ there exist positive numbers h, C such that $|F(x, \xi)| \le C \exp(h |\xi|^\rho)$ for $(x, \xi) \in \Omega'$.

A formal sum $\sum_{j=0}^\infty P_j(x, \xi)$ of symbols $P_j(x, \xi)$ defined in Ω is said to be a formal symbol of growth order at most (ρ) if for each compactly generated cone $\Omega'\subset\Omega$ there are positive numbers C, A, h such that

(2.1) $$|P_j(x,\xi)|\le CA^j j!\,|\xi|^{-j}\exp(h|\xi|^\rho)$$

for $(x,\xi)\in\Omega'$, $j=0,1,2,\cdots$ (cf. [2], [3]). If $\sum_{j=0}^\infty P_j(x,\xi)$ is a formal symbol of growth order at most (ρ), then the sum of the operators $\sum_{j=0}^\infty :P_j(x,\xi):$ converges to an operator P of growth order at most (ρ) in $\mathcal{E}^R_{\mathring{x}^*}$. Then it may be written, for convenience, as $P=:\sum_{j=0}^\infty P_j(x,\xi):$ even if the sum $\sum_{j=0}^\infty P_j(x,\xi)$ of holomorphic functions does not converge.

Let $P=:\sum P_j(x,\xi):$ and $Q=:\sum Q_k(x,\xi):$ be operators defined by formal symbols. Then the composite operator $R=PQ$ is expressed as $R=:\sum R_l(x,\xi):$ with formal symbol defined by

(2.2) $$R_l(x,\xi)=\sum_{j+k+|\alpha|=l}\frac{1}{\alpha!}\partial_\xi^\alpha P_j(x,\xi)\cdot\partial_x^\alpha Q_k(x,\xi),$$

for $l=0,1,2,\cdots$. Remark that we distinguish ∂_x, etc. from D_x, etc.; for example, $D_{x_1}f-fD_{x_1}=\partial_{x_1}f$, where f is a holomorphic function.

We refer the reader to [2] for the precise definition and basic properties of symbols of holomorphic microlocal operators.

Now let Ω be a conic neighborhood of $\mathring{x}^*$ in T^*X. Let $p(x,\xi)$ and $q(x,\xi)$ be symbols defined in Ω satisfying the following conditions. For each compactly generated cone $\Omega'\subset\Omega$ there exist positive numbers h, H such that

(2.3) $$\begin{cases}|p(x,\xi)|\le h|\xi|^\rho+H,\\ |q(x,\xi)|\le h|\xi|^\rho+H,\end{cases}$$

for $(x,\xi)\in\Omega'$. Take the symbols $\exp\{p(x,\xi)\}$ and $\exp\{q(x,\xi)\}$. It is clear that they are symbols of growth order at most (ρ). Let us consider the composite operator $:\exp\{p(x,\xi)\}:\ :\exp\{q(x,\xi)\}:$ of $:\exp\{p(x,\xi)\}:$ and $:\exp\{q(x,\xi)\}:$. It is plausible that the leading term of the symbols of the composite operator may be the product of each symbol, that is, $\exp\{p(x,\xi)+q(x,\xi)\}$. The following theorem shows, however, that it is not always true.

Theorem 1. *Assume that* $0\le\rho\le 1/2$. *Then we have the following exponential rules.*

(2.4) $$:\exp\{p(x,\xi)\}::\exp\{q(x,\xi)\}: \;=\; :\exp\{p(x,\xi)+q(x,\xi)\}\sum_{j=0}^\infty K_j(x,\xi):.$$

Here $\sum_{j=0}^\infty K_j(x,\xi)$ *is a formal symbol of growth order* (0) *satisfying the following.*

(2.5) $$K_0(x,\xi)=\exp\left\{\sum_{i=1}^n\partial_{\xi_i}p(x,\xi)\cdot\partial_{x_i}q(x,\xi)\right\}+\tilde K_0(x,\xi),$$

(2.6) *For each* $\Omega'\subset\Omega$ *there is* $C_1>0$ *such that*
$$|\tilde K_0(x,\xi)|\le C_1|\xi|^{-1/2},\qquad (x,\xi)\in\Omega'.$$

Remark that $C_2^{-1}\le|K_0(x,\xi)|\le C_2$ in Ω' for some $C_2>0$ because $0\le\rho$

$\leq 1/2$. If $0\leq\rho<1/2$, then it follows from the preceding theorem that the leading term of the symbol of the composite operator is $\exp\{p(x,\xi)+q(x,\xi)\}$.

Moreover we have the following

Theorem 2. *Suppose that $0\leq\rho\leq 2/3$. Then we have the exponential laws of the form*

$$(2.7)\qquad :\exp\{p(x,\xi)\}:\ :\exp\{q(x,\xi)\}: \\ =\ :\exp\{p(x,\xi)+q(x,\xi)+r_1(x,\xi)\}\sum_{j=0}^{\infty}H_j(x,\xi):.$$

Here

$$(2.8)\qquad r_1(x,\xi)=\sum_{i=1}^{n}\partial_{\xi_i}p(x,\xi)\cdot\partial_{x_i}q(x,\xi),$$

and $\sum_{j=0}^{\infty}H_j(x,\xi)$ is a formal symbol of growth order (0) *satisfying*

$$(2.9)\qquad H_0(x,\xi)=\exp\{r_2(x,\xi)\}+\tilde{H}_0(x,\xi),$$

$$(2.10)\qquad r_2=\frac{1}{2}\sum_{\mu,\nu=1}^{n}(\partial_{\xi_\mu}\partial_{\xi_\nu}p\cdot\partial_{x_\mu}q\cdot\partial_{x_\nu}q+\partial_{\xi_\mu}p\cdot\partial_{\xi_\nu}p\cdot\partial_{x_\mu}\partial_{x_\nu}q),$$

(2.11) *For each $\Omega'\subset\Omega$ there is $C_3>0$ such that $|\tilde{H}_0(x,\xi)|\leq C_3|\xi|^{-1/3}$, $(x,\xi)\in\Omega'$.*

We also remark that $C_4^{-1}\leq|H_0(x,\xi)|\leq C_4$ in Ω' for some $C_4>0$ since $0\leq\rho\leq 2/3$.

The following theorem is an improvement on Theorem 3.3.1 in [2] where the growth order is assumed to be at most (1/2).

Theorem 3. *Let P be a holomorphic microlocal operator of growth order at most* (2/3) *defined in a neighborhood of $\mathring{x}^*$. Suppose that the symbol $P(x,\xi)$ of P is invertible as a symbol of growth order at most* (2/3) *in some neighborhood of $\mathring{x}^*$. That is, assume that $1/P(x,\xi)$ is also a symbol of growth order at most* (2/3). *Then P is invertible in the ring $\mathcal{E}^R_{\mathring{x}^*}$. The inverse U of P is of the form*

$$(2.12)\qquad U=\ :\exp\{-p(x,\xi)\}:\ :\exp\left\{\sum_{i=1}^{n}\partial_{\xi_i}p(x,\xi)\cdot\partial_{x_i}p(x,\xi)\right\}:R.$$

Here $p(x,\xi)=\log P(x,\xi)$ and R is an operator of growth order at most (0). *The leading term of the symbol of R is*

$$(2.13)\qquad \exp\left\{\frac{1}{2}\sum_{\mu,\nu=1}^{n}(\partial_{\xi_\mu}p\cdot\partial_{\xi_\nu}p\cdot\partial_{x_\mu}\partial_{x_\nu}p-\partial_{\xi_\mu}\partial_{\xi_\nu}p\cdot\partial_{x_\mu}p\cdot\partial_{x_\nu}p)\right\}.$$

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