111. A Note on the Convolution Theorem on Functions of Several Variables

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1. J. Mikusiński and C. Ryll-Nardzewski [1], [2] generalized the well-known Titchmarsh convolution theorem, which plays a fundamental roll in Mikusiński's operational calculus [3], to functions of several variables; see K. Yosida and S. Okamoto [4] for their operational calculus without appealing to the Titchmarsh convolution theorem. Mikusiński and Ryll-Nardzewski showed:

Theorem. *Let* $T>0$, *and* $\omega \in \boldsymbol{R}_+^n$; $\boldsymbol{R}_+^n=\{x=(x_1, \cdots, x_n); x_j>0 \ (j=1, \cdots, n)\}$. *Let* f, g *be two integrable functions on* S_T; $S_T=\{x \in \boldsymbol{R}_+^n; 0<x\cdot\omega<T\}$, $(x\cdot\omega=\sum_{j=1}^n x_j\omega_j)$. *If the convolution of* f *and* g *vanishes almost everywhere* (a.e.) *in* S_T:

$$(1) \qquad f*g(x)=\int_0^{x_1}\cdots\int_0^{x_n} f(x-y)g(y)dy_1\cdots dy_n=0, \text{ a.e. in } S_T$$

$(x=(x_1, \cdots, x_n); y=(y_1, \cdots, y_n))$, *then there are two non-negative numbers* T_1, T_2 *with* $T_1+T_2\geq T$ *such that* $f(x)=0$, a.e. *in* S_{T_1} *and* $g(x)=0$, a.e. *in* S_{T_2}.

This theorem was first formulated and proved by E. Titchmarsh [5], [6] for $n=1$. The original proof was based on a difficult theorem in the complex analytic function theory; see M. Crum [7] and J. Dufresnoy [8] for simpler proofs.

Elementary proofs, based exclusively on the method of functions of a real variable, were given by Mikusiński [9], and Mikusiński and Ryll-Nardzewski [3] by making use of the Lerch moment theorem. Mikusiński and Ryll-Nardzewski extended the Titchmarsh convolution theorem to functions of several variables by a geometrical method. J. Lions [10], [11] obtained similar results by making use of the Fourier transform, assuming that the functions considered satisfy some growth conditions of exponential type. The purpose of the present note is to show the Titchmarsh-Mikusiński-Ryll·Nardzewski theorem by making use of the Fourier transform without imposing any such restriction on functions. Prof. K. Yosida kindly communicated to the author another proof ([12]) which makes use of the change of variables in the double integration, the Liouville theorem and the Weierstrass polynomial approximation theorem when the present work was almost completed.

2. Proof of the theorem. *The first step.* Let $c>0$. Let H be the totality of all continuous functions h on S_{2c} with $h*h=0$ in S_{2c}. In this step we shall show that $h=0$ in S_c if $h\in H$. For $h\in H$, we set

$$h_1(x)=h(x) \qquad (\text{for } x \text{ in } S_c)\,;=0 \text{ (elsewhere)}.$$

We then claim that the entire function $\hat{h}_1(z\omega)$ of a complex variable z satisfies the estimate:

$$\hat{h}_1(z\omega)=O(e^{c\,\mathrm{Im}\,z})^{1)}, \qquad z\in \boldsymbol{C} \tag{2}$$

where $\hat{f}(\zeta)$ is the Fourier-Laplace transform of f:

$$\hat{f}(\zeta)=(2\pi)^{-n/2}\int_{R^n} e^{-i\zeta\cdot x}f(x)dx.$$

By a simple calculation, we have $(0\leq a<b<\infty)$

$$\int_{S_{a,b}} e^{-iz\omega\cdot x}h(x)dx=\begin{cases}O(e^{a\,\mathrm{Im}\,z}) & (\text{for Im } z<0)\\ O(e^{b\,\mathrm{Im}\,z}) & (\text{for Im } z\geq 0)\end{cases} \tag{3}$$

where $S_{a,b}=\{x\in R^n_+\,;\,a\leq x\cdot\omega\leq b\}$. If $\mathrm{Im}\,z\geq 0$, then (2) follows immediately from (3). We shall show (2) for $\mathrm{Im}\,z<0$. Set

$$h_2(x)=h(x) \qquad (\text{for } x \text{ in } S_{c,2c})\,;=0 \text{ (elsewhere)}$$

and

$$h_3(x)=h_1(x)+h_2(x).$$

By (3), $\hat{h}_2(z\omega)=O(e^{c\,\mathrm{Im}\,z})$ ($\mathrm{Im}\,z<0$). Since $h_3(x)=h(x)$ for x in S_{2c}, we have, by $h\in H$,

$$h_3*h_3(x)=h*h(x)=0 \qquad \text{in } S_{2c}.$$

Hence the support of h_3*h_3 is contained in $S_{2c,4c}$. Hence by (3),

$$(\hat{h}_3(z\omega))^2=\widehat{h_3*h_3}(z\omega)=O(e^{2c\,\mathrm{Im}\,z})$$

and so $\hat{h}_3(z\omega)=O(e^{c\,\mathrm{Im}\,z})$ (for $\mathrm{Im}\,z<0$). Therefore by $h_1=h_3-h_2$,

$$|\hat{h}_1(z\omega)|\leq|\hat{h}_3(z\omega)|+|\hat{h}_2(z\omega)|=O(e^{c\,\mathrm{Im}\,z}),$$

showing (2) for $\mathrm{Im}\,z<0$.

Now the entire function $e^{icz}\hat{h}_1(z\omega)$ of z is, by (2), bounded and so is constant by the Liouville theorem. Hence

$$\int_{S_c} e^{-izx\cdot\omega}(c-\omega\cdot x)h(x)dx=i^{-1}e^{-icz}\frac{d}{dz}(e^{izc}\hat{h}_1(z\omega))=0 \tag{4}$$

for all $h\in H$. If $h\in H$, then $e^{i\zeta\cdot x}h(x)\in H$ for all $\zeta\in\boldsymbol{C}^n$. Hence by (4) with $z=0$,

$$\int_{S_c}(c-\omega\cdot x)e^{i\zeta\cdot x}h(x)dx=0$$

for all $\zeta\in\boldsymbol{C}^n$ and $h\in H$. By the uniqueness of the Fourier transform, $(c-\omega\cdot x)h(x)=0$, and so $h(x)=0$ in S_c.

The second step. In this step, following Mikusiński [13] we shall show: If f, g are continuous functions on S_T with $f*g=0$ in S_T, then for any $\zeta\in\boldsymbol{C}^n$,

$$f*g_k=0 \qquad \text{in } S_T\ (k=1,2,\cdots) \tag{5}$$

where $g_k(x)=(\zeta\cdot x)^k g(x)$. Indeed, let α be the largest number $(0\leq\alpha\leq 1)$

1) "$f(z)=O(g(z))$" means $f(z)/g(z)$ is bounded for large $|z|$.

such that

$$(6)\qquad f*g_1=0 \quad \text{in } S_{\alpha T}$$

for any continuous functions f, g with $f*g=0$ in S_T. It is easy to see that such an α does not depend on T. Hence,

$$(7)\qquad f_1*g_1=0 \quad \text{in } S_{\alpha^2 T}.$$

Since $f*g=0$ in S_T, we have

$$(8)\qquad f_1*g+f*g_1=0 \quad \text{in } S_T.$$

On the other hand, we can easily see: Let u, v be continuous functions. Then,

$$(9)\qquad u*v=0 \quad \text{in } S_{a+b}$$

if $u=0$ in S_a and $v=0$ in S_b $(a, b>0)$. Hence, by (6), (8), and (9)

$$f*g_1*(f_1*g+f*g_1)=0 \quad \text{in } S_{T+\alpha T}$$

and so, by the commutativity of the convolution,

$$(f*g_1)*(f*g_1)+(f*g)*(f_1*g_1)=0 \quad \text{in } S_{T+\alpha T};$$

extend appropriately the functions to the interval $[0, T+\alpha T]$. By (1), (7), (9), the second term on the left side vanishes in $S_{T+\alpha^2 T}$. By the first step, $f*g_1=0$ in $S_{(\alpha^2+1)T/2}$. Hence $(\alpha^2+1)/2\leq\alpha$ by the definition of α, which implies $\alpha=1$. This proves (5) for $k=1$. Inductively we can obtain (5) for $k=2, \cdots$.

The third step. If f and g are continuous and satisfy (1) in S_T, then by the second step

$$\int_0^{x_1}\cdots\int_0^{x_n} f(x-y)\,e^{i\zeta\cdot y}g(y)dy$$
$$=\sum_{k=0}^{\infty}\frac{(i)^k}{k!}\int_0^{x_1}\cdots\int_0^{x_n} f(x-y)(\zeta\cdot y)^k g(y)dy=0$$

for $x\in S_T$ and $\zeta\in \boldsymbol{C}^n$. By the uniqueness of the Fourier transform,

$$(10)\qquad f(x-y)g(y)=0$$

for $x\in S_T$ and $y=(y_1, \cdots, y_n)$ with $0\leq y_j\leq x_j$ $(j=1, \cdots, n)$. Let T_1 (resp. T_2) the largest number such that f (resp. g) vanishes identically in S_{T_1} (resp. S_{T_2}). Suppose that $T_1+T_2<T$. Then there must be points x_1, x_2 (in R_+^n) with $x_1\cdot\omega+x_2\cdot\omega<T$ such that $f(x_1)\neq 0$, $g(x_2)\neq 0$. Take $x=x_1+x_2$, and $y=x_2$ in (10). Then $f(x_1)g(x_2)=0$, which is a contradiction. This proves the theorem for the case f, g are continuous. If f, g are integrable, put $f_0=1*f$, $g_0=1*g$. Then it is easy to see that f_0, g_0 are continuous and satisfy (1). Hence $f_0=0$ in S_{T_1} and $g_0=0$ in S_{T_2} $(T_1+T_2\geq T)$. Differentiating f_0 (resp. g_0) with respect to x in S_{T_1} (resp. S_{T_2}), we get the desired result. This completes the proof of the theorem.

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