110. Nonlinear Rotative Semigroups

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(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1982)

Let $(E, |\cdot|)$ be a Banach space, and let I denote the identity operator. Recall that a subset A of $E\times E$ with domain $D(A)$ and range $R(A)$ is said to be *accretive* if $|x_1-x_2|\le|x_1-x_2+r(y_1-y_2)|$ for all $[x_i, y_i]\in A$, $i=1,2$, and $r>0$. The resolvent $J_r: R(I+rA)\to D(A)$ of A is defined by $J_r=(I+rA)^{-1}$, and its Yosida approximation A_r by $A_r=(I-J_r)/r$. We denote the closure of a subset D of E by $\mathrm{cl}\,(D)$ and its distance from the origin by $\|D\|$. We shall say that A satisfies the *range condition* if $R(I+rA)\supset \mathrm{cl}(D(A))$ for all $r>0$. In this case, $-A$ generates a (nonexpansive) *nonlinear semigroup* $S:[0,\infty)x\in\mathrm{cl}\,(D(A))\to\mathrm{cl}\,(D(A))$ by the exponential formula [3]:

$$S(t)x=\lim_{n\to\infty}(I+(t/n)A)^{-n}x.$$

We shall say that S is *rotative* (or, more precisely, p-rotative) if for some $p>0$ and $a<1$,

$$|x-S(p)x|\le ap\|Ax\|$$

for all $x\in D(A)$. This definition is analogous to that of [4], where a nonexpansive mapping $T: D\to D$ is said to be rotative if there exist an integer $m\ge 2$ and a real number $a<1$ such that $|x-T^m x|\le am|x-Tx|$ for all x in D. As a matter of fact, we show below that if $A=I-T$ satisfies the range condition and S is the semigroup generated by $-A$, then S is rotative if and only if T is. Examples of rotative nonexpansive mappings include strict contractions in any Banach space and rotations in any Euclidean space. Periodic semigroups, as well as those semigroups with strongly accretive (negative) generators, are also rotative. Note that if S is an arbitrary nonlinear semigroup, then $|x-S(t)x|\le t\|Ax\|$ for all $t\ge 0$ and $x\in D(A)$. It follows that if S is p-rotative, then it is also q-rotative for all $q\ge p$ (possibly with a larger $a<1$).

Our purpose in this note is to introduce nonlinear rotative semigroups and to establish some of their properties. Our main result is that rotative semigroups with a convex domain always have a fixed point. We emphasize that in contrast with other fixed point theorems

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for nonlinear semigroups (see [2] for example), our result holds in all Banach spaces.

We begin by establishing the connection between rotative mappings and rotative semigroups.

Proposition 1. *Let D be a closed subset of a Banach space and $T: D \to D$ a nonexpansive mapping. Assume that $A = I - T$ satisfies the range condition and let S be the semigroup generated by $-A$. Then S is rotative if and only if T is.*

Proof. Assume that T is m-rotative, and let $k \geq 1$ be an integer. Since $|S(n)x - T^n x| \leq n^{1/2}|x - Tx|$ for all $n \geq 1$ and $x \in D$ [5, 1, and 7, p. 82], we see that

$$\begin{aligned} |x - S(km)x| &\leq |x - T^{km}x| + |T^{km}x - S(km)x| \leq (kam + (km)^{1/2})|x - Tx| \\ &= (a + (km)^{-1/2})(km)|x - Tx| \end{aligned}$$

for some $a < 1$. Hence S is (km)-rotative for sufficiently large k. Conversely, we may assume that S is m-rotative for some integer m. Therefore $|x - T^{km}x| \leq |x - S(km)x| + |S(km)x - T^{km}x| \leq (kam + (km)^{1/2})$ $|x - Tx|$, and the result follows.

Proposition 2. *Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and S the semigroup generated by $-A$. If S is rotative, then $0 \in \mathrm{cl}\,(R(A))$.*

Proof. Denote $\|R(A)\|$ by d. It is known [8, Proposition 3.1] that $\lim_{t \to \infty} |S(t)x/t| = d$ for each x in $\mathrm{cl}\,(D(A))$. If S is p-rotative and $k \geq 1$ is an integer, then $|x - S(kp)x| \leq kap\|Ax\|$ for some $a < 1$ and all $x \in D(A)$. Dividing both sides of this inequality by kp and letting $k \to \infty$, we obtain $d \leq a\|Ax\|$ for all $x \in D(A)$. Thus $d \leq ad$ and $d = 0$.

Our next result shows that if $\mathrm{cl}\,(D(A))$ is convex, then in fact $0 \in R(A)$.

Theorem. *Let E be a Banach space, $A \subset E \times E$ an accretive operator with a convex $\mathrm{cl}\,(D(A))$ that satisfies the range condition, and S the semigroup generated by $-A$. If S is rotative, then it has a fixed point.*

Proof. Fix a positive r and let J_r be the resolvent $(I + rA)^{-1}$ of A. We may assume that S is (mr)-rotative for some integer $m \geq 1$. We then have, for $y \in D(A)$ and integer $k \geq 1$, $|y - J_r^{km}y| \leq |y - S(kmr)y| + |S(kmr)y - J_r^{km}y| \leq kamr\|Ay\| + (km)^{1/2}r\|Ay\|$. Applying this inequality to $y = J_r x$, we obtain $|J_r x - J_r^{km+1}x| \leq (kamr + (km)^{1/2}r)\|AJ_r x\| \leq (kamr + (km)^{1/2}r)|A_r x| = (kam + (km)^{1/2})|x - J_r x|$. Consequently,

$$|x - J_r^{km+1}x| \leq (1 + kam + (km)^{1/2})|x - J_r x|$$

for all $x \in \mathrm{cl}\,(D(A))$. Thus $J_r : \mathrm{cl}\,(D(A)) \to \mathrm{cl}\,(D(A))$ is $(km+1)$-rotative for sufficiently large k, and has a fixed point by [4]. This fixed point is also a fixed point of S.

Note that if $T: D \to D$ is a rotative nonexpansive mapping, then

$I-T$ need not, in general, satisfy the range condition, nor need T have a fixed point. If D is convex, then $I-T$ always satisfies the range condition, and a rotative T has a fixed point.

In the course of the proof of Theorem, we have shown that if a rotative S is generated by $-A$, then the resolvent J_r of A is rotative for each $r>0$. As a matter of fact, the converse is also true.

Proposition 3. *Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and S the semigroup generated by $-A$. Then S is rotative if and only if J_r is rotative for some (hence all) $r>0$.*

Proof. Suppose that J_r is m-rotative for some $r>0$ and let $k \geq 1$ be an integer. Then

$$\begin{aligned} |x-S(kmr)x| &\leq |x-J_r^{km}x|+|J_r^{km}x-S(kmr)x| \\ &\leq kam\,|x-J_rx|+(km)^{1/2}r\,\|Ax\| \\ &= kamr\,|A_rx|+(km)^{1/2}r\,\|Ax\| \\ &\leq (kamr+(km)^{1/2}r)\,\|Ax\|. \end{aligned}$$

Thus S is (kmr)-rotative for sufficiently large k.

We conclude with the observation that Proposition 3 is applicable in the context of product formulas.

Proposition 4. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, and C a closed convex subset of E. Let $F(t): C \to C$, $0 \leq t < \infty$, be a continuous family of nonexpansive mappings with $F(0)=I$. Assume that $\lim_{n\to\infty} F(t/n)^n x = S(t)x$ exists for each $x \in C$ uniformly on compact t intervals, and that each $F(t)$ is $m(t)$-rotative (with the same $a<1$), where $m(t) \leq M/t$ as $t \to 0+$. Then the semigroup S is also rotative.*

Proof. We know [6, p. 157] that $J_r x = \lim_{t\to 0+} (I+(r/t)(I-F(t)))^{-1}x$ exists and is the resolvent of an accretive operator that is the (negative) generator of S. Since J_r can be shown to be rotative, so is S by Proposition 3.

Acknowledgement. We are grateful to Professor K. Goebel for several helpful conversations.

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