108. *Existence of Global Solutions for Nonlinear Wave Equations*

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§1. **Statement of result.** In this paper we consider nonlinear wave equations of the following type:

(1.1) $\Phi[u]\equiv\square u+F(u, Du, D_xDu)=0,\qquad \text{for } t\in[0,\infty),\quad x\in \mathrm{R}^n,$

with the initial conditions:

(1.2) $u(0, x)=\varphi(x),$

(1.3) $\dfrac{\partial u}{\partial t}(0, x)=\psi(x),\qquad \text{for } x\in\mathrm{R}^n.$

Here the symbols D_x and D denote $(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ and $(\partial/\partial t, D_x)$ respectively, and $\square$ denotes the wave operator $(\partial/\partial t)^2-\sum_{i=1}^n(\partial/\partial x_i)^2$. We sometimes use the variable x_0 in place of t. The function F of (1.1) is a function of variables $\xi=(\lambda; \lambda_i, i=0, \cdots, n; \lambda_{ij}, i, j=0, \cdots, \mathrm{n}, i+j>0)$ and it is of class C^∞ in a neighborhood of the origin $\xi=0$. Moreover we assume that

(A) $F(0)=\dfrac{\partial F}{\partial \xi}(0)=0.$

The initial data φ and ψ are supposed to belong to

$$W_2^\infty(\mathrm{R}^n)=\bigcap_{m=0}^\infty \mathrm{W}_2^m(\mathrm{R}^n),$$

here $W_2^m(\mathrm{R}^n)$ is the Sobolev space of order m.

Then our result is

Theorem. *Suppose that the space dimension n is greater than or equal to* 12 *and that the condition A is satisfied. Then there exist an integer N and a small constant $\eta>0$ such that for any initial data satisfying*

$$\|\varphi\|_{L_1,N}+\|\psi\|_{L_1,N}<\eta \quad \textit{and} \quad \|\varphi\|_{L_2,N}+\|\psi\|_{L_2,N}<\eta,$$

the problem (1.1)–(1.3) *has a unique solution in* $C^\infty([0,\infty)\times\mathrm{R}^n)$.

Remark. Our problem differs from that of Klainerman [3] in the point that our F depends on λ as well as λ_i, λ_{ij}. The essential difference appears in the energy estimates for the linearized problems which play an important role in the iteration process.

Hereafter we use following norms for a function $f(t, x)$ defined on $[0,\infty)\times\mathrm{R}^n$.

$$\|f\|_{L_p,m}(t)=\|f(t,\cdot)\|_{L_p,m};$$
$$|f|_m(t)=\|f\|_{L_\infty,m}(t) \quad\text{and}\quad \|f\|_m(t)=\|f\|_{L_2,m}(t),$$

moreover

$$\|f\|_{k,L_p,m}=\sup_{t\in[0,\infty)}(1+t)^k\|f\|_{L_p,m}(t);$$
$$|f|_{k,m}=\|f\|_{k,L_\infty,m} \quad \text{and} \quad \|f\|_{k,m}=\|f\|_{k,L_2,m}.$$

We introduce some abbreviations.

$$\tilde{\xi}=(\lambda;\ \lambda_i,\ i=0,\cdots,n;\ \lambda_{ij},\ i,j=0,\cdots,n,\ i+j>0),$$
$$\xi=(\tilde{\xi},\ \lambda_{00}),$$
$$\tilde{\Xi}=(1;\ D_i,\ i=0,\cdots,n;\ D_{ij},\ i,j=0,\cdots,n,\ i+j>0),$$
$$\Xi=(\tilde{\Xi},\ D_{00}),$$

and for example a differential operator $b_\xi(t,x)\cdot\tilde{\Xi}$ denotes

$$\sum_{\substack{i,j=0,\cdots,n\\ i+j>0}} b_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=0}^n b_i(t,x)\frac{\partial}{\partial x_i}+b(t,x).$$

§ 2. **Energy estimates.** We derive the energy estimates for solutions of the linear hyperbolic equations:

$$\Box v+b_\xi(t,x)\cdot\tilde{\Xi}v=g(t,x), \qquad \text{for } t\in[0,\infty),\quad x\in\mathbf{R}^n,$$
$$v(0,x)=\frac{\partial v}{\partial t}(0,x)=0, \qquad \text{for } x\in\mathbf{R}^n.$$

Proposition. *Suppose that* $v\in C^2([0,\infty),\ W_2^\infty(\mathbf{R}^n))$, $g\in C^0([0,\infty),\ W_2^\infty(\mathbf{R}^n))$ *and* $b_\xi\in C^1([0,\infty),\ \mathscr{B}(\mathbf{R}^n))$. *Moreover we make the following assumptions:*

(2.1) $$|b_\xi|_{0,0}<3/(4n),$$

(2.2) $$|b_\xi|_{2+\varepsilon,0},\quad |b_\xi|_{1+\varepsilon,1} \quad \textit{and} \quad \left|\frac{\partial}{\partial t}b_\xi\right|_{1+\varepsilon,0}<1,$$

for some positive constant ε. *Then we have*

(2.3) $$\|Dv\|_0(t)\le C(n,\varepsilon)\int_0^t\|g\|_0(\tau)d\tau,$$

(2.4) $$\|Dv\|_1(t)\le C(n,\varepsilon)\int_0^t\{\|g\|_1(\tau)+|b_\xi|_1(\tau)\|v\|_0(\tau)\}d\tau,$$

and

(2.5) $$\|Dv\|_m(t)\le C(m,n,\varepsilon)\int_0^t\{\|g\|_m(\tau)+|b_\xi|_m(\tau)\|\tilde{\Xi}v\|_0(\tau)\}d\tau,$$

for $m\ge 2$.

Remark. Our condition about the space dimension is more restrictive than that of [3]. It is because of the following facts.

(a) The condition (2.2) is more severe than the similar one in [3].

(3) We can not evaluate $\|v\|_{0,0}$ because we evaluate $\|v\|_0(t)$ by the integral $\int_0^t\|\partial v/\partial t\|_0(\tau)d\tau$.

Sketch of the proof of Proposition. Let $E(v)$ be the energy of v, i.e.,

$$\{E(v)\}^2=\int_{\mathbf{R}^n}\left\{\left(\frac{\partial v}{\partial t}\right)^2+\sum_{\substack{i,j=0,\cdots,n\\ i+j>0}}(\delta_{ij}-b_{ij})\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right\}dx.$$

Then by the standard method we have the following estimate:

(2.6) $$E(v) \leq C(n)\Big[\|g\|_0(t) + \Big\{|b_\xi|_1(t) + \Big|\frac{\partial}{\partial t} b_\xi\Big|_0(t)\Big\}E(v) + |b_\xi|_0 \int_0^t E(v)(\tau)d\tau\Big].$$

In order to evaluate $E(v)$ we use the following

Lemma. *Suppose that α, β and γ are C^0 functions defined on $[0, \infty)$ and are non-negative and that a C^1 function f satisfies the following inequality:*

$$\frac{df}{dt}(t) \leq \alpha(t) + \beta(t)\cdot f(t) + \gamma(t)\int_0^t f(\tau)d\tau, \qquad \text{for } t>0,$$

$$f(0)=0.$$

Then

$$f \leq \int_0^t \alpha(\tau)d\tau \cdot \exp\Big[\int_0^t \{\beta(\tau) + \tau\cdot\gamma(\tau)\}d\tau\Big].$$

We omit the proof of Lemma and continue the proof of Proposition. Applying the lemma to the inequality (2.6), we have

$$\begin{aligned} E(v) &\leq C(n)\int_0^t \|g\|_0(\tau)d\tau \\ &\quad \times \exp\Big[C(n)\int_0^t \Big\{|b_\xi|_1(\tau) + \Big|\frac{\partial}{\partial t} b_\xi\Big|_0(\tau) + \tau\cdot|b_\xi|_0(\tau)\Big\}d\tau\Big] \\ &\leq C(n)\int_0^t \|g\|_0(\tau)d\tau \\ &\quad \times \exp\Big[C(n)\int_0^t \Big\{\frac{|b_\xi|_{1+\varepsilon,1}}{(1+\tau)^{1+\varepsilon}} + \frac{|(\partial/\partial t)b_\xi|_{1+\varepsilon,0}}{(1+\tau)^{1+\varepsilon}} + \frac{\tau|b_\xi|_{2+\varepsilon,0}}{(1+\tau)^{2+\varepsilon}}\Big\}d\tau\Big]. \end{aligned}$$

Condition (2.2) yields that $E(v) \leq C(n, \varepsilon)\int_0^t \|g\|_0(\tau)d\tau$. We obtain (2.3) since $\|Dv\|_0(t)$ is equivalent to the energy $E(v)(t)$. This equivalence follows from the condition (2.1). Estimates (2.4) and (2.5) can be derived by usual arguments.

§3. Outline of the proof of Theorem. We make the solution by the iteration method and use the same iteration scheme as that of [3]. First we review it briefly.

The 0th approximation u_0 is the solution of the following.

$$\Box u_0 = 0, \qquad \text{for } t \in [0, \infty),\ x \in \mathbf{R}^n,$$

$$u_0(0, x) = \varphi(x),\ \frac{\partial}{\partial t} u_0(0, x) = \psi(x), \qquad \text{for } x \in \mathbf{R}^n.$$

Suppose u_j, $j=0, \cdots, p$ and $\dot{u}_j$, $j=0, \cdots, p-1$ are defined. Then we define $\dot{u}_p$ as the solution of the following equation:

(3.1) $$\begin{aligned} &\Box \dot{u}_p + \frac{\partial F}{\partial \xi}(S_p\tilde{\Xi} u_p)\cdot\tilde{\Xi}\dot{u}_p = g_p, \qquad \text{for } t \in [0, \infty),\ x \in \mathbf{R}^n, \\ &\dot{u}_p(0, x) = \frac{\partial}{\partial t}\dot{u}_p(0, x) = 0, \qquad \text{for } x \in \mathbf{R}^n, \end{aligned}$$

where S_j, $j=0, 1, 2, \cdots$ are the smoothing operators used in [3] and the functions g_j, $j=0, 1, 2, \cdots$ are defined as follows.

$$g_j=-S_je_{j-1}-(S_j-S_{j-1})\Big(\sum_{i=0}^{j-2}e_i+\Phi[u_0]\Big),$$
$$e_j=F(\tilde{\Xi}u_{j+1})-F(\tilde{\Xi}u_j)-\frac{\partial F}{\partial\xi}(S_j\tilde{\Xi}u_j)\cdot\tilde{\Xi}\dot{u}_j.$$

We define u_{p+1} to be $u_p+\dot{u}_p$.

We derive inductively the following estimates for each $p=0,1,2,\cdots$.

(3.2. p) $$|\Xi\dot{u}_p|_{k,m}\leq\delta\theta_p^{k-\beta+\varepsilon m},\quad \text{for } 0\leq k\leq\tilde{k},\ 0\leq m\leq\tilde{m},$$
$$\|\Xi\dot{u}_p\|_{-1,m}\leq\delta\theta_p^{-1-\beta+\varepsilon m},\quad \text{for } 0\leq m\leq\tilde{m}.$$

Here $\tilde{k}=(n-1)/2$, $\theta_p=2^p$ and δ is a small positive constant depending on the function F. And $\tilde{m}$, ε and β are constants satisfying the following inequalities:

(3.3) $$\tilde{k}-1-2\beta\geq\varepsilon,$$

(3.4) $$-2\beta+\varepsilon\tilde{m}\geq\varepsilon,$$

(3.5) $$\tilde{k}\geq 3+\beta+\varepsilon\Big(\Big[\frac{n}{2}\Big]+2\Big),$$

(3.6) $$\beta\geq 2+\varepsilon\Big(\Big[\frac{n}{2}\Big]+2\Big).$$

Remark. We use these inequalities in the proof of (3.2). And such constants exist if and only if the space dimension n is greater than or equal to 12.

Once the estimate (3.2) is obtained, it is easy to check that the series $\sum_{p=0}^{\infty}\dot{u}_p$ converges in the space $C^2([0,\infty)\times\mathbf{R}^n)$ and that the function $u=u_0+\sum_{p=0}^{\infty}\dot{u}_p$ is the solution of the problem (1.1)–(1.3). So we only give the outline of the proof of (3.2).

Suppose (3.2.j), $j=0,\cdots,p$ hold. Then using (3.3) and (3.4), we can prove

$$|g_{p+1}|_{k,m}<C\delta^2\theta_{p+1}^{k-2\beta+\varepsilon m},\quad \text{for } 0\leq k,\ 0\leq m,$$
$$\|g_{p+1}\|_{k,m}<C\delta^2\theta_{p+1}^{k-2\beta+\varepsilon m},\quad \text{for } 0\leq k,\ 0\leq m,$$
$$\|g_{p+1}\|_{-1,L_1,m}<C\delta^2\theta_{p+1}^{-1-\beta+\varepsilon m},\quad \text{for } 0\leq m,\ -1-\beta+\varepsilon m\geq\varepsilon.$$

We apply the energy estimates in § 2 to the problem (3.1). Then the following L_2 estimates are obtained.

(3.7) $$\|\Xi\dot{u}_{p+1}\|_{-1,m}<C\delta^2\theta_{p+1}^{1-2\beta+\varepsilon m},\quad \text{for } 0\leq m.$$

Since $\beta\geq 2$, the second part of (3.2.p+1) follows from (3.7). The decay estimates in [3] and the inequalities (3.5) and (3.6) yield the first part of (3.2.p+1). Therefore we obtain (3.2.p+1).

References

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