106. *Smooth Global Solutions for the One-Dimensional Equations in Magnetohydrodynamics*

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§ 1. **Introduction.** The motion of electrically conducting fluids on one space coordinate is described by the equations in Lagrangian representation ([1]):

$$(1)\quad \begin{cases}(1/\rho)_t-u_x=0,\\ u_t+(p+|B|^2/2\mu_0)_x=(\nu\rho u_x)_x, \qquad v_t-(\bar{B}^1B/\mu_0)_x=(\mu\rho v_x)_x,\\ \theta_t+(\theta p_\theta/e_\theta)u_x=(1/e_\theta)\{(\kappa\rho\theta_x)_x+\nu\rho u_x^2+\mu\rho|v_x|^2+(\rho/\sigma\mu_0^2)|B_x|^2\},\\ (B/\rho)_t-(\bar{B}^1v)_x=\{(\rho/\sigma\mu_0)B_x\}_x.\end{cases}$$

Here $\rho>0$, $\boldsymbol{u}=(u^1,u^2,u^3)\in\boldsymbol{R}^3$, $\theta>0$ and $\boldsymbol{B}=(\bar{B}^1,B^2,B^3)\in\boldsymbol{R}^3$ represent the mass density, the velocity, the absolute temperature and the magnetic induction, where we write $u=u^1$, $v=(u^2,u^3)$, $B=(B^2,B^3)$, and $\bar{B}^1$ is a constant.

We assume that the pressure p and the internal energy e are smoothly related to ρ and θ by the equations of state

$$(2)\qquad p_\rho>0,\quad e_\theta>0,\quad de=\theta dS-pd(1/\rho),$$

where $S=S(\rho,\theta)$ is the entropy; the coefficients of viscosity μ,ν, the coefficient of heat conductivity κ and the coefficient of electrical resistivity $1/\sigma$ (σ: the coefficient of electrical conductivity) are all smooth functions of ρ and θ, and are positive or identically zero; μ_0 is the magnetic permeability, now a positive constant.

In this paper, we seek smooth solutions of (1) in a small neighborhood of a constant state $(\rho,\boldsymbol{u},\theta,B)=(\bar{\rho},0,\bar{\theta},\bar{B})$ where $\bar{\rho}>0$, $\bar{\theta}>0$ and $\bar{B}\in\boldsymbol{R}^2$ are arbitrary fixed constants. To obtain the a priori estimates for the solutions, we use the following energy form ([4]):

$$\mathcal{E}=e-\bar{e}+\bar{p}(1/\rho-1/\bar{\rho})-\bar{\theta}(S-\bar{S})+|\boldsymbol{u}|^2/2+|B-\bar{B}|^2/2\mu_0\rho,$$

where $\bar{e}=e(\bar{\rho},\bar{\theta})$ and so on. Note that if $|\rho-\bar{\rho},\theta-\bar{\theta}|$ is small, $\mathcal{E}$ is equivalent to the quadratic form $|\rho-\bar{\rho},\boldsymbol{u},\theta-\bar{\theta},B-\bar{B}|^2$. This is based on the strict convexity of the internal energy e as a function of $1/\rho$ and S.

From (1) and (2), we have the energy conservation law:

$$(3)\quad \begin{aligned}&(e+|\boldsymbol{u}|^2/2+|B|^2/2\mu_0\rho)_t+\{(p+|B|^2/2\mu_0)u-(\bar{B}^1B/\mu_0)\cdot v\}_x\\ &\quad=\{\nu\rho uu_x+\mu\rho v\cdot v_x+\kappa\rho\theta_x+(\rho/\sigma\mu_0^2)B\cdot B_x\}_x,\end{aligned}$$

and the equation of entropy:

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(4) $S_t=\{(\kappa\rho/\theta)\theta_x\}_x+(1/\theta)\{\nu\rho u_x^2+\mu\rho|v_x|^2+(\kappa\rho/\theta)\theta_x^2+(\rho/\sigma\mu_0^2)|B_x|^2\}$.

Using (1), (3) and (4), we have the identity for $\mathcal{E}$ which plays an important role in the present paper:

(5) $$\begin{aligned}&\mathcal{E}_t+\{(p+|B|^2/2\mu_0-\bar{p}-|\bar{B}|^2/2\mu_0)u-(\bar{B}^1(B-\bar{B})/\mu_0)\cdot v\}_x\\&\quad+(\bar{\theta}/\theta)\{\nu\rho u_x^2+\mu\rho|v_x|^2+(\kappa\rho/\theta)\theta_x^2+(\rho/\sigma\mu_0^2)|B_x|^2\}\\&=\{\nu\rho uu_x+\mu\rho v\cdot v_x+(1-\bar{\theta}/\theta)\kappa\rho\theta_x+(\rho/\sigma\mu_0^2)(B-\bar{B})\cdot B_x\}_x.\end{aligned}$$

§ 2. **Results and remarks.** We consider the system (1) with the initial data:

(6) $$\begin{cases}(\rho, \boldsymbol{u}, \theta, B)(0, x)=(\rho_0, \boldsymbol{u}_0, \theta_0, B_0)(x), & x\in \boldsymbol{R}^1,\\ \inf\{\rho_0(x),\ \theta_0(x)\,;\ x\in \boldsymbol{R}^1\}>0,\end{cases}$$

where $\boldsymbol{u}_0=(u_0, v_0)$. We set up two cases: $0<\sigma<\infty$ (finitely conducting) and $\sigma=\infty$ (perfectly conducting). In each case there are the following four cases:

(i) $\mu, \nu, \kappa>0$, (ii) $\mu=\nu=0$, $\kappa>0$,

(iii) $\mu, \nu>0$, $\kappa=0$, (iv) $\mu=\nu=\kappa=0$.

Theorem 1. *Let us assume that* $0<\sigma<\infty$, $\bar{B}^1\neq 0$ *and one of the above cases* (i)–(iv) *for the system* (1); *in the cases* (ii) *and* (iv) *we also assume the additional conditions* $|p_\theta(\bar{\rho}, \bar{\theta})|+|\bar{B}|\neq 0$ *and* $|\bar{B}|\neq 0$, *respectively. Moreover assume that* $(\rho_0-\bar{\rho}, \boldsymbol{u}_0, \theta_0-\bar{\theta}, B_0-\bar{B})\in H^2(\boldsymbol{R}^1)$ *for the initial data* (6). *Then if* $\|\rho_0-\bar{\rho}, \boldsymbol{u}_0, \theta_0-\bar{\theta}, B_0-\bar{B}\|_2$ *is sufficiently small, the initial value problem* (1) (6) *has a unique smooth solution* $(\rho, \boldsymbol{u}, \theta, B)(t, x)$ *global in time. Here* $\|\cdot\|_l$ *denotes the norm of the Sobolev space* $H^l(\boldsymbol{R}^1)$.

Remarks. 1. In the cases (i) (ii) the solution $(\rho, \boldsymbol{u}, \theta, B)(t)$ converges to the constant state $(\bar{\rho}, 0, \bar{\theta}, \bar{B})$ as $t\to\infty$ in the maximum norm. While in the cases (iii) (iv) we only know; $(p(\rho, \theta), \boldsymbol{u}, B)(t)$ approaches to $(p(\bar{\rho}, \bar{\theta}), 0, \bar{B})$ as $t\to\infty$ in the maximum norm. 2. If $\bar{B}^1=0$ is assumed for (1), in every case of (i)–(iv) we also establish the same results as above ones except that the equation of v becomes trivial ($v(t, x)=v_0(x)$) for (ii) or (iv). 3. When all the coefficients μ, ν, κ and $1/\sigma$ are positive and independent of θ, we can show the existence of a classical global solution of (1) in the Hölder space ([4]).

Neglecting the magnetic field and the second and third components of the velocity ($\boldsymbol{B}=v=0$) in (1), we have the usual system in fluid dynamics:

(7) $$\begin{cases}(1/\rho)_t-u_x=0, \qquad u_t+p_x=(\nu\rho u_x)_x,\\ \theta_t+(\theta p_\theta/e_\theta)u_x=(1/e_\theta)\{(\kappa\rho\theta_x)_x+\nu\rho u_x^2\}.\end{cases}$$

For the system (7), statements in Theorem 1 are simplified as follows:

Corollary. *For* (7) *we assume one of the cases* (i)–(iii) (*for case* (ii) *we assume* $p_\theta(\bar{\rho}, \bar{\theta})\neq 0$ *in addition*). *Then if* $\|\rho_0-\bar{\rho}, u_0, \theta_0-\bar{\theta}\|_2$ *is approprеately small, a unique smooth solution* $(\rho, u, \theta)(t, x)$ *of* (7) *exists for all time.*

Remark. It is well known that in the case (iv) smooth solutions of (7) in general develop singularities in the first derivatives in finite time ([2]).

Theorem 2. *Assume that* $\sigma=\infty$, $\bar{B}^1\neq 0$ *and* (i) *or* (iii) *for* (1). *Then if the initial data are small as in Theorem* 1, *a unique smooth solution of* (1) *exists globally in time.*

Remarks. 1. In spite of $\sigma=\infty$, we can show the same decay law as in Theorem 1. 2. When (iv) is satisfied then the first derivatives of solutions of (1) become infinite in finite time ([2]), while in the case of (ii) we have no results on the existence or non-existence of global solutions of (1). 3. If $\bar{B}^1=0$, the last equation of (1) implies $(B/\rho)(t,x)=(B_0/\rho_0)(x)$. Therefore the system (1) is reduced to

$$\begin{cases}(1/\rho)_t-u_x=0, \qquad u_t+\{p+(1/2\mu_0)|B_0/\rho_0|^2\rho^2\}_x=(\nu\rho u_x)_x,\\ \theta_t+(\theta p_\theta/e_\theta)u_x=(1/e_\theta)\{(\kappa\rho\theta_x)_x+\nu\rho u_x^2\},\end{cases}$$

where we set $v=0$ for simplicity. In this system, it seems that additional considerations are necessary to the general case of $B_0/\rho_0 \neq \text{constant}$.

§3. Proof of theorems. Since local existence theorem is well known ([5]), to show the existence of a global solution, it suffices to obtain the a priori estimates for the solution. We prove the estimates only in the case that $0<\sigma<\infty$, $\bar{B}^1\neq 0$ and (iv). The method here is also applicable with slight modification to the other cases and gives the analogous estimates.

Set $Q_T=[0,T]\times \boldsymbol{R}^1$ (for $T>0$) and

$$E(T)^2=\sup\{\|(\rho-\bar{\rho},\boldsymbol{u},\theta-\bar{\theta},B-\bar{B})(t)\|_2^2;\ t\in[0,T]\}+\int_0^T\|D_x(p(\rho,\theta),\boldsymbol{u})(\tau)\|_1^2+\|D_xB(\tau)\|_2^2\,d\tau.$$

Lemma (a priori estimate). *Let T be some positive constant. Assume that $(\rho,\boldsymbol{u},\theta,B)(t,x)$ satisfies* $\inf\{\rho(t,x),\theta(t,x);(t,x)\in Q_T\}>0$ *and $E(T)<\infty$, and is a solution of* (1) *in the case that $0<\sigma<\infty$, $\bar{B}^1\neq 0$ and* (iv) $(|\bar{B}|\neq 0)$. *Then if $E(T)$ is suitably small, we have the a priori estimate $E(T)\le CE(0)$ for some constant $C>1$ independent of T.*

Proof. Integrating (5) with $\mu=\nu=\kappa=0$ over Q_t $(t\in[0,T])$, we have the following $L^2(\boldsymbol{R}^1)$-estimate for the solution:

$$\|(\rho-\bar{\rho},\boldsymbol{u},\theta-\bar{\theta},B-\bar{B})(t)\|^2+\int_0^t\|D_xB(\tau)\|^2\,d\tau\le CE(0)^2, \tag{8}$$

where $\|\cdot\|$ denotes the $L^2(\boldsymbol{R}^1)$-norm. Next, in the same way as [3], we obtain the $L^2(\boldsymbol{R}^1)$-estimates for the derivatives of the solution. Rewrite the system (1) with $\mu=\nu=\kappa=0$ by the change of variables:

$$\begin{cases}p_t+qu_x=(\rho p_\theta/e_\theta\sigma\mu_0^2)|B_x|^2, \qquad u_t+(p+|B|^2/2\mu_0)_x=0,\\ v_t-(\bar{B}^1B/\mu_0)_x=0, \qquad S_t=(\rho/\theta\sigma\mu_0^2)|B_x|^2,\\ B_t+\rho(Bu_x-\bar{B}^1v_x)=\rho\{(\rho/\sigma\mu_0)B_x\}_x,\end{cases} \tag{9}$$

where $q=\rho^2p_\rho+\theta p_\theta^2/e_\theta>0$ by (2).

Operate $D_x^l=(\partial/\partial x)^l$, $l=0,1,2$, on each equation in (9) and we obtain the system of $D_x^l(p, \boldsymbol{u}, S, B)$. First, multiplying the equations of $D_x^l p, D_x^l(\boldsymbol{u}, S)$ and $D_x^l B$ by $(1/q)D_x^l p, D_x^l(\boldsymbol{u}, S)$ and $(1/\mu_0\bar{\rho})D_x^l B$ respectively, summing them up, integrating over Q_t, and then adding the resulting equality for $l=1,2$, we obtain after integration by parts

$$(10)\qquad \|D_x(\rho, \boldsymbol{u}, \theta, B)(t)\|_1^2+\int_0^t \|D_x^2 B(\tau)\|_1^2\, d\tau \leq C(E(0)^2+E(T)^3).$$

Secondly, multiply the equations of $D_x^l p$, $D_x^l u$ and $D_x^l B$ by $2(\bar{B}^1\bar{B}/q)\cdot D_x^l v_x$, $\beta D_x^l p_x$ (β: positive constant) and $(\bar{B}D_x^l u_x-\bar{B}^1 D_x^l v_x)/\bar{\rho}$ respectively, sum up the equalities obtained and integrate it over Q_t. Estimating the resulting equality by use of the Schwarz inequality, taking β suitably small and adding for $l=0,1$, we arrive at

$$(11)\qquad \int_0^t \|D_x(p, \boldsymbol{u})(\tau)\|_1^2\, d\tau - C\{\|(\rho-\bar{\rho}, \boldsymbol{u}, \theta-\bar{\theta}, B-\bar{B})(t)\|_2^2 + \int_0^t \|D_x B(\tau)\|_2^2\, d\tau\} \leq C(E(0)^2+E(T)^3),$$

where $(\bar{B}^1, \bar{B})\neq 0$ is used.

Joining (8), (10) and (11) together, we gain the inequality $E(T)^2 \leq C(E(0)^2+E(T)^3)$ from which the assertion in Lemma follows directly. This completes the proof.

Finally we show the asymptotic behavior of the solution. Since $D_x(p, \boldsymbol{u}) \in L^2(0, \infty; H^1(\boldsymbol{R}^1))$ and $D_x B \in L^2(0, \infty; H^2(\boldsymbol{R}^1))$, we have $\partial_t(p, \boldsymbol{u}, B) \in L^2(0, \infty; H^1(\boldsymbol{R}^1))$ by use of (9). Therefore we conclude that $\|D_x(p, \boldsymbol{u}, B)(t)\| \to 0$ as $t\to\infty$. It follows from this that

$$\sup\{|(p(\rho, \theta)-p(\bar{\rho}, \bar{\theta}), \boldsymbol{u}, B-\bar{B})(t)|\,;\, x\in \boldsymbol{R}^1\}$$

converges to zero as $t\to\infty$ with the aid of the Sobolev inequality in one space dimension.

This completes the proof of Theorems.

References

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