102. *Siegel Modular Forms of Degree Two*

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Introduction. In this note, we discuss a correspondence between the space of modular forms of half integral weight and the space of Siegel modular forms of degree two, and its application to Maass spaces, in close relation with Saito-Kurokawa's conjecture (cf. [2], [3], [4], [8]).

Let M be any positive integer, χ a character mod M, $\tilde{M}=\text{l.c.m}(4, M)$, and k an even integer. In our previous paper [3], we constructed a linear mapping $\Psi_k^{M,\chi}$ of $\mathfrak{S}_{2k-1}(\tilde{M},\chi)$ into $S_k(\Gamma_0^{(2)}(M),\bar{\chi})$. In this note, we construct another linear mapping Ψ of $\mathfrak{S}_{2k-1}(4N,\chi)$ into $S_k(\Gamma_0^{(2)}(2N),\chi)$, k being an even integer and χ a character mod $2N$. It will be seen that Ψ is more useful than $\Psi_k^{M,\chi}$ in several points and serves to generalize our results in [3]. For example, Theorem 4 in [3] is generalized in the sense that the assumption (5.1) in [3] can be dropped.

§ 1. We denote by $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the field of real numbers and the field of complex numbers. For a ring A, we denote by A_m^n the set of all $n\times m$ matrices with entries in A, and denote A_1^n (resp. A_n^n) by A^n (resp. $M_n(A)$). For a $z\in\boldsymbol{C}$, we set $e[z]=\exp(2\pi iz)$ with $i=\sqrt{-1}$ and we define $\sqrt{z}=z^{1/2}$ so that $-\pi/2<\arg(z^{1/2})\leqq\pi/2$. Denote by $\mathfrak{H}_n$ the complex Siegel upper half space of degree n. Let $Sp(n,\boldsymbol{R})$ be the real symplectic group of degree n. For a positive integer N, we consider a congruence subgroup of the Siegel modular group of degree n defined by

$$\Gamma_0^{(n)}(N)=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix}\in M_{2n}(\boldsymbol{Z})\cap Sp(n,\boldsymbol{R})\,|\,C\equiv 0 \pmod N\right\}\quad (\Gamma_0^{(1)}(N)=\Gamma_0(N)).$$

We denote by $S_k(\Gamma_0^{(2)}(M),\psi)$ the space of Siegel modular cusp forms F of Neben-type ψ and of weight k with respect to $\Gamma_0^{(n)}(M)$ satisfying

$$F((AZ+B)(CZ+D)^{-1})=(\bar{\psi}\det(A))(\det(CZ+D))^k F(Z)$$
$$\left(Z\in\mathfrak{H}_n,\ \begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_0^{(n)}(M)\right).$$

We consider two symmetric matrices

$$Q_0=\begin{pmatrix}S&0&0\\0&S&0\\0&0&-1\end{pmatrix}\quad\text{and}\quad Q_1=\begin{pmatrix}S&0\\0&-1\end{pmatrix}\left(S=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right).$$

For a positive integer N, set $L(N)=\{{}^t(x_1,x_2,2Nx_3,(1/N)x_4,\sqrt{2}\,x_5)\,|\,x_i\in\boldsymbol{Z}\}$

and $L'(N)=\{{}^t(x_1, x_2, x_3, (1/N)x_4, \sqrt{2}x_5) \mid x_i \in Z\}$. Let ρ be the isomorphism of $Sp(2, R)/\{\pm id\}$ onto $O(Q_0)_0=\{g \in M_5(R) \mid {}^t g Q_0 g = Q_0\}_0$ given in [2]. For a character χ modulo $2N$, we define a theta series $\Theta^N_{k,\chi}(z, g)$ on $\mathfrak{H}_1 \times Sp(2, \boldsymbol{R})$ by

$$\Theta^N_{k,\chi}(z, g) = v^{(3-k)/2} \sum_{l \in L'(N)/L(N)} \chi(l_3) \sum_{h \in L(N)} e[uN^t(l+h)Q_0(l+h)/2] f_k(\sqrt{v}\, \rho(g)^{-1}(l+h)),$$

where $z=u+iv \in \mathfrak{H}_1$, $l={}^t(\cdots, l_3, \cdots)$ and $f_k(x)=(N^t x Q_0(-i, i, 1, -1, 0))^k \times \exp(-\pi N^t xx)(x \in \boldsymbol{R}^5)$.

§ 2. We denote by $\mathfrak{S}_{2k-1}(4N, \chi)$ the space of modular cusp forms of Neben-type χ and of weight $(2k-1)/2$ with respect to $\Gamma_0(4N)$ (cf. [6]). For an $f(z)=\sum_{n=1}^{\infty} a(n)e[nz] \in \mathfrak{S}_{2k-1}(4N, \chi)$, we define a function $\Psi(f)(Z)$ on $\mathfrak{H}_2$ by

$$\Psi(f)(Z)=(\det(Ci+D))^k \int_{\Gamma_0(4N)\backslash\mathfrak{H}_1} v^{(2k-1)/2} f \mid [W_{4N}]_{2k-1}(z) \overline{\Theta^N_{k,\bar{\chi}}(z, g)} \frac{dudv}{v^2}$$

with $Z=(Ai+B)(Ci+D)^{-1}$, $z=u+iv$ and $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \boldsymbol{R})$, where $f \mid [W_{4N}]_{2k-1}(z)=f(-1/4Nz)(4N^{1/4}(-iz)^{1/2})^{-(2k-1)}$ and $\Gamma_0(4N)\backslash\mathfrak{H}_1$ is a fundamental domain for $\Gamma_0(4N)$. We can verify the following theorem in the same fashion as in [3].

Theorem 1. *Suppose that χ is a character modulo $2N$ and k (>5) is even. Then $\Psi(f)$ belongs to $S_k(\Gamma_0^{(2)}(2N), \chi)$ and the Fourier expansion $\Psi(f)$ has the form*

$$\Psi(f)(Z)=\tilde{c} \sum_T \sum_{\substack{d \mid e(T) \\ d>0}} \chi(d) d^{k-1} a(N(T)/d^2) e[\mathrm{tr}(TZ)],$$

where $\tilde{c}$ is a non-zero constant not depending upon f, T runs over all symmetric positive semi-definite semi-integral matrices of size 2,

$$e(T)=\text{g.c.m.}(m, t, n) \quad \textit{and} \quad N(T)=4\det(T) \quad \left(T=\begin{pmatrix} m & t/2 \\ t/2 & n \end{pmatrix}\right).$$

§ 3. We denote by $T^{4N}_{2k-1,\chi}(p^2)$ (resp. $\tilde{T}^{2N}_{k,\chi}(n)$) Hecke operators defined on $\mathfrak{S}_{2k-1}(4N, \chi)$ (resp. $S_k(\Gamma_0^{(2)}(2N), \chi)$) (cf. [1], [5], [6] and [7]). We can also prove the following

Theorem 2. *With k and χ as in Theorem 1, let f be an element of $\mathfrak{S}_{2k-1}(4N, \chi)$ such that $T^{4N}_{2k-1,\chi}(p^2)f=\omega(p)f$ for all primes p. Then $\Psi(f)$ is an eigenfunction of Hecke operators $\tilde{T}^{2N}_{k,\chi}(n)$ for all positive integers n. Moreover, the zeta function attached to $\Psi(f)$ in the sense of Andrianov coincides with*

$$L(s-k+1, \chi)L(s-k+2, \chi) \prod_p (1-\omega(p)p^{-s}+\chi(p)^2 p^{2k-3-2s})^{-1},$$

where $L(s, \chi)=\sum_{n=1}^{\infty} \chi(n)n^{-s}$.

We call a form $F(Z)=\sum_T a(T)e[\mathrm{tr}(TZ)] \in S_k(\Gamma_0^{(2)}(M), \psi)$ a Maass form, if Fourier coefficients $a(T)$ satisfy

$$a\left(\begin{pmatrix} m & t/2 \\ t/2 & n \end{pmatrix}\right)=\sum_{\substack{d \mid (m,t,n) \\ d>0}} \psi(d) d^{k-1} \times a\left(\begin{pmatrix} 1 & t/2d \\ t/2d & mn/d^2 \end{pmatrix}\right)$$

(cf. [4]). Let $f(z)=\sum_{n=1}^{\infty} a(n)e[nz](a(1)\neq 0)$

$$\left(\text{resp. } F(Z)=\sum_{T} a(T)e[\mathrm{tr}\,(TZ)]\left(a\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\neq 0\right)\right)$$

be a cusp form of $\mathfrak{S}_{2k-1}(4N,\chi)$ (resp. a Maass form of $S_k(\Gamma_0^{(2)}(2N),\chi)$) such that $T_{2k-1,\chi}^{4N}(p^2)f=\omega(p)f$ (resp. $\tilde{T}_{k,\chi}^{2N}(n)F=\lambda(n)F$) for all primes p (resp. positive integers n). We denote then by $\tilde{\mathfrak{S}}_{2k-1}(4N,\chi)$ (resp. $\mathcal{M}_k(\Gamma_0^{(2)}(2N),\chi)$) the vector space spanned by all such elements f (resp. F). The following theorem can be proved in the same manner as in [3].

Theorem 3. *Let N be a positive integer satisfying $(2,N)=1$, and let χ be a character modulo $2N$. Then ψ induces a linear isomorphic mapping between $\tilde{\mathfrak{S}}_{2k-1}(4N,\chi)$ and $\mathcal{M}_k(\Gamma_0^{(2)}(2N),\chi)$.*

This theorem is a generalization of Theorem 4 in [3].

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