9. On Eisenstein Series of Degree Two for Hilbert-Siegel Modular Groups

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Introduction. In this note we present an explicit formula for Fourier coefficients of generalized Eisenstein series of degree two for Hilbert-Siegel modular groups in the sense of Langlands [8] and Klingen [4]. This explicit formula is a generalization of the previous result in [7] [11] (the Siegel modular case), and has an application to the algebraicity of the special value of the "second" L-function attached to a Hilbert modular form. Details will appear elsewhere. The author would like to thank Prof. N. Kurokawa for suggestions and encouragements.

§ 1. Generalized Eisenstein series for Hilbert-Siegel modular groups. Let F be a totally real number field of degree \( g \) over Q, \( \mathcal{O}_F \) the ring of integers in F, \( E = \mathcal{O}_F^\times \) the group of units in F, and \( E_+ = \{ \varepsilon \in E | \varepsilon > 0 \} \) the group of totally positive units in F. Let \( F^{(i)}, \ldots, F^{(g)} \) be the conjugates of F over Q with \( F^{(i)} = F \). The image of an element \( \lambda \in F \) (resp. a matrix \( M \) with all entries lying in F) under \( F \to F^{(i)} \) is denoted by \( \lambda^{(i)} \) (resp. \( M^{(i)} \)). If \( \lambda^{(i)} > 0 \) (resp. \( M = M^{(i)} \) and \( M^{(i)} > 0 \); \( M = M^{(i)} \) and \( M^{(i)} > 0 \)) \( \lambda > 0 \) (resp. \( M > 0 \); \( M > 0 \)).

For a positive integer \( n \), we put \( \Gamma^{(i)}_{\mathfrak{n}} = \{ \Gamma \in M_n(\mathcal{O}_F) | \varepsilon \mathcal{J} \mathcal{M} = \varepsilon \mathcal{J} \} \) for some \( \varepsilon \in E_+ \) where \( \mathcal{J}_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \) (\( E_n \) is the identity matrix of size \( n \)). For an integer \( k \geq 0 \), we denote by \( M_k(\Gamma^{(i)}_{\mathfrak{n}}) \) (resp. \( S_k(\Gamma^{(i)}_{\mathfrak{n}}) \)) the \( \mathbb{C} \)-vector space of all Hilbert-Siegel modular (resp. cusp) forms of weight \( k \) with respect to \( \Gamma^{(i)}_{\mathfrak{n}} \). We denote by \( E_k(\Gamma^{(i)}_{\mathfrak{n}}) \) the orthogonal complement of \( S_k(\Gamma^{(i)}_{\mathfrak{n}}) \) in \( M_k(\Gamma^{(i)}_{\mathfrak{n}}) \) with respect to the Petersson inner product. As usual we put \( M_k(\Gamma^{(i)}_{\mathfrak{n}}) = S_k(\Gamma^{(i)}_{\mathfrak{n}}) = \mathbb{C} \).

Let \( n, k, r \) be integers such that \( n \geq 1, 0 \leq r \leq n, k > n + r + 1 \). We assume the following condition:

(a) \( k \) is an even integer if \( F \) contains a unit with norm \( -1 \).

Generalizing the construction of Klingen [4], we put

\[
E_{k,n}^{\mathfrak{n}}(Z, f) = \sum_{M \in \mathcal{A}_{n,r}(\Gamma^{(i)}_{\mathfrak{n}})} f(M(Z)^k)N_{\mathfrak{f}/\mathfrak{q}}((CZ + D)^{-k})^{\mathcal{R}} \text{ for } f \in S_k(\Gamma^{(i)}_{\mathfrak{n}}).
\]

Here \( \mathcal{A}_{n,r} \) is the subgroup of \( \Gamma^{(i)}_{\mathfrak{n}} \) of all \( M \in \Gamma^{(i)}_{\mathfrak{n}} \) whose entries in the first \( n + r \) columns and last \( n - r \) rows vanish, and \( M = (\begin{pmatrix} A & B \\ C & D \end{pmatrix})(A, B, C, D \) are square matrices of size \( n \)\) runs over a complete system of repre-
sentatives of the left cosets of $\Gamma^{(p)}$ modulo $A_n$; $Z=(Z_1, \ldots, Z_g)$ is a variable on $\mathfrak{H}_g$, the product of $g$-copies of the Siegel upper half space of degree $n$; for each $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(p)}$ we put $M\langle Z \rangle^* = (M^{(1)} \langle Z \rangle^*, \ldots, M^{(g)} \langle Z \rangle^*)$ where $M^{(i)} \langle Z \rangle^* = (A^{(i)} Z_1 + B^{(i)})(C^{(i)} Z_1 + D^{(i)})^{-1}$ and $M^{(i)} \langle Z \rangle^*$ is the square matrix formed by the first $(r, r)$-entries of $M^{(i)} \langle Z \rangle$; and $N_{r, q}(q C Z + D) = \prod_{1 \leq i \leq g} |C^{(i)} Z_1 + D^{(i)}|$, $| \det(\cdot) |$ denoting the determinant. Then the above summation defining $E_{n, r}(Z, f)$ converges uniformly and absolutely on $\{(Z_1, \ldots, Z_g) \in \mathfrak{H}_g | \text{trace}(X_i) \leq c^{-1}, Y_i \geq c E_n (1 \leq i \leq g) \}$ for any $c > 0$ and represents an element of $M(F)$. Moreover $\Phi E_{n, r}(\ast, f) = E_{n-1, r}(\ast, f)$ for $r < n$ and $\Phi E_{n, n}(\ast, f) = \Phi f = 0$ where $\Phi$ is the Siegel operator. (For definitions, we refer to Christian [1]). Results in Klingen [4] are generalized to the Hilbert-Siegel case as follows:

Proposition. Let $n, k$ be integers such that $k > 2n \geq 0$, and suppose that $k$ satisfies the condition (a). Put $E_k(\Gamma^{(p)}) = \{E_{n, r}(\ast, f) | f \in S_k(\Gamma^{(p)}) \}$ for $0 \leq r \leq n$. Then:

1. $M_k(\Gamma^{(p)}) = \oplus_{0 \leq r \leq n} E_{k}(\Gamma^{(p)})$, $E_k(\Gamma^{(p)}) = \oplus_{0 \leq r \leq n-1} E_k(\Gamma^{(p)})$, and $S_k(\Gamma^{(p)}) = E_{k}(\Gamma^{(p)})$.

2. $\Phi$ induces a $C$-linear isomorphism: $E_k(\Gamma^{(p)}) \approx E_{k-1}(\Gamma^{(p-1)})$ for each $r = 0, \ldots, n-1$ if $n \geq 1$.

§ 2. An explicit formula of Fourier coefficients for degree two case. Throughout this section we assume the following conditions:
(b) $F$ is a totally real number field with the class number one in the narrow sense, and (c) the rational prime $2$ decomposes completely in $F$. Let $k > 0$ be an integer satisfying the condition (a) in § 1. Let $f \in M_k(\Gamma^{(p)})$ be a normalized eigen Hilbert modular form in the following sense: $f(z) = \sum_{a \in \mathcal{O}_F} a(\ell) \phi(z, \ell)$ with $a(\mathcal{O}_F) = 1$ and $T(m) f = a(m) f$ for all Hecke operators $T(m)$ associated with integral ideals $m$ of $F$. Here, $\phi(z) = \exp(2\pi \sqrt{-1} z)$, $z = (z_1, \ldots, z_g) \in \mathfrak{H}_g$, $T_{r, \mathcal{O}_F}(\ell)$ is $\sum_{1 \leq i \leq g} \ell_i(z_i)$, and $\delta$ is the different of $F / Q$. By the assumption (b), $\delta = (\delta)$ with $\delta > 0$. As in [5] [6] [7] [11], we put $[f] = E_{k, \mathcal{O}_F}(\ast, f)$ if $\Phi f = 0$ and $[f] = E_{k, \mathcal{O}_F}(\ast, f)$ if $\Phi f \neq 0$. Let $[f](Z) = \sum_{r \geq 0} a(T, [f]) \phi(\sigma(\mathcal{O}_F)(\delta^{-1} T Z))$ be the Fourier expansion of $[f]$, where $T$ runs over all symmetric totally positive semi-definite semi-integral (i.e. $T = (t_{ij})$, $2t_{ij} \in \mathcal{O}_F$, $t_{ii} \in \mathcal{O}_F$ for $1 \leq i, j \leq 2$) matrices of size 2, and $\sigma$ is the trace of matrices. To obtain a formula for $a(T, [f])$, it is sufficient to consider the case $T \gg 0$. We denote by $d(F)$ the discriminant of $F$.

Theorem 1. For $T \gg 0$ such that $|2T|$ is square-free (i.e., $|2T|$ is not divisible by the square of any proper ideal in $\mathcal{O}_F$), we have:

$$a(T, [f]) = \frac{1}{2} (-1)^{k/2} \left( 2(2\pi)^{k-1} \frac{(k-1)!}{(2k-2)!} \right)$$
Here \( g = (F : Q) \), \( \chi \) denotes the Hecke character attached to the quadratic extension \( F(\sqrt{-2T})/F \), \( L_p(s, \chi) \) the Hecke \( L \)-function, and \( \vartheta_{T}(z) \) = \( \sum_{\xi \in \mathbb{Z}^{2}} \sum_{(a, b) \in \mathbb{Z}^{2} \times \mathbb{Q}} e^{i\pi(a^2b)} \). We take complex numbers \( \alpha, \beta \) such that \( \sum_{\alpha, \beta} a(a)N(\alpha)^{-\epsilon} = \prod_{p}(1-\alpha N(p)^{-\epsilon}(1-\beta N(p)^{-\epsilon})^{-1} \), where \( \alpha \) runs over all integral ideals of \( F \) and \( \beta \) runs over all prime ideals of \( F \), and put \( L_{\chi}(s, f) = \prod_{p}(1-\alpha N(p)^{-\epsilon}(1-\beta N(p)^{-\epsilon})^{-1} \). Writing \( \vartheta_{T}(z) = \sum_{b \in \mathbb{Z}} b((\lambda)b)e^{i\pi(bz)} \), we put \( D(s, f, \vartheta_{T}) = \sum_{a} a(a)b(a)N(\alpha)^{-\epsilon} \). Each \( L \)-function is considered as a meromorphic function on \( C \) by the analytic continuation. If \( \vartheta_{T}(z) \neq 0 \), then \( D(s, f, \vartheta_{T}) \) and \( L_{\chi}(2s, f) \) have zeros of the same order at \( s = k-1 \), and we understand that \( D(k-1, f, \vartheta_{T})/L(2k-2, f) \) if \( \vartheta_{T}(z) \neq 0 \).
Then, as in Kurokawa [6], Harris [3], and Garrett [2], we have \( a(T, [f]) \in \overline{\mathbb{Q}} \) for all \( T > 0 \), where \( \overline{\mathbb{Q}} \) denotes the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \).
(The author received Garrett's preprint [2] in October 1981 after the first draft of this paper was prepared.)
Moreover we know by Theorem 1 (after a short argument) that for any \( T > 0 \) with \(|2T|\) square-free integer in \( \mathcal{O}_\mathbb{F} \) there exists some \( T_1 > 0 \) such that \(|2T_1|=|2T|\) and that \( a(T_1, [f]) \neq 0 \). Hence, by Theorem 1 and Remark 2 combined with a result of Shimura [12] on \( D(k-1, f, h) \), we have the following:

**Theorem 3.** Let \( F, g, k, \) and \( f \) be as above. Then:
\[
L_s(2k-2, f)/\pi^{(2k-2)/2} \langle f, f \rangle \in \overline{\mathbb{Q}}.
\]
Here, \( \langle f, f \rangle \) denotes the normalized Petersson norm, i.e. 
\[
\langle f, f \rangle = \operatorname{vol}(\mathfrak{F})^{-1} \int_{\mathfrak{F}} |f(z)|^2 \operatorname{Im}(z)^s d\mu(z),
\]
where \( \mathfrak{F} \) is a fundamental domain of \( \mathcal{F}_g \), \( \operatorname{Im}(z) = \prod_{1 \leq i \leq g} y_i \) and \( d\mu(z) = \prod_{1 \leq i \leq g} y_i^{-2} dx_i dy_i \) if \( z = (z_1, \ldots, z_g) \), 
\( z_i = x_i + \sqrt{-1} y_i \).

**References**


