90. On Homotopy Self-Equivalences of the Product $A\times B$

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§ 1. **Introduction.** The set of homotopy classes of self-homotopy equivalences of a CW-complex X, which is denoted by $G(X)$, forms a group with the multiplication defined by the composition of maps. This group $G(X)$ has been studied by many authors since M. Arkowitz and C. R. Curjel's paper [1] was published in 1964. In particular, the group $G(A\times B)$ was considered by A. J. Sieradski in [4] for two connected H-spaces A and B, by N. Sawashita in [3] for the case of a product of spheres $S^m\times S^n$; also S. Sasao and one of the authors studied the group $G(K(\pi,1)\times X)$ in [2] for a simply-connected CW-complex X. The purpose of this paper is to investigate the group $G(A\times B)$, and especially to generalize the results of [2].

Throughout this paper we use the following notations. For two based CW-complexes (X, x_0) and (Y, y_0), we denote by X^Y the space of all continuous based maps from Y to X with the compact open topology, and by $[Y, y_0; X, x_0]=[Y, X]=\pi_0(X^Y)$ the set of path-components of X^Y. If G is a monoid, $\mathrm{Inv}\,(G)$ denotes the group consisting of invertible elements of G. Let $\mathrm{pr}_A: A\times B\to A$ and $\mathrm{pr}_B: A\times B\to B$ be the natural projections to the first factor and to the second factor, respectively.

Our main theorem states:

Theorem. *Let A and B be CW-complexes satisfying*

(a) $[B, A]=[A\wedge B, A]=\{0\}$, *and*

(b) *B is simply-connected.*

Then there is a split exact sequence:

$$1\longrightarrow \mathrm{Inv}\,([A, a_0; B^B, \mathrm{id}_B])\longrightarrow G(A\times B)\longrightarrow G(A)\times G(B)\longrightarrow 1.$$

Corollary. *Under the same assumptions as Theorem,*

$$G(A\times B)\simeq G(A)\times G(B) \qquad \textit{if } \mathrm{Inv}\,([A, a_0; B^B, \mathrm{id}_B])=1.$$

Example. If B is simply-connected, there is a split exact sequence:

$$1\longrightarrow \pi_2(B^B, \mathrm{id}_B)\sharp(\pi_1(B^B, \mathrm{id}_B)\times\pi_1(B^B, \mathrm{id}_B))\longrightarrow G(T^2\times B)$$
$$\longrightarrow GL_2(Z)\times G(B)\longrightarrow 1,$$

where T^2 denotes the two dimensional torus $S^1\times S^1$ and $\sharp$ denotes a semi-direct sum.

Remark. If B is an H-space or a co H-space, then we have $\pi_1(B^B, \mathrm{id}_B)\simeq[SB, B]$.

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§ 2. **Lemmas.** We define the multiplication $\times : A^{A\times B}\times A^{A\times B}\to A^{A\times B}$ by

$$(f\times g)(a,b)=f(g(a,b),b) \qquad \text{for } f,g\in A^{A\times B} \text{ and } (a,b)\in A\times B.$$

Then it is easy to see that $(A^{A\times B},\times)$ is a monoid with the unit pr_A. Similarly, we define the multiplication $\otimes : (A^{A\times B}\times B^{A\times B})^2\to A^{A\times B}\times B^{A\times B}$ by

$$f\otimes g=(f_1(g_1,g_2),f_2(g_1,g_2)) \qquad \text{for } f=(f_1,f_2),\ g=(g_1,g_2)\in A^{A\times B}\times B^{A\times B}.$$

Then it is also easy to see that $(A^{A\times B}\times B^{A\times B},\otimes)$ is a monoid with the unit $(\mathrm{pr}_A,\mathrm{pr}_B)$. Let $\eta:(A\times B)^{A\times B}\to A^{A\times B}\times B^{A\times B}$ be the natural homeomorphism defined by

$$\eta(f)=(\mathrm{pr}_A\circ f,\mathrm{pr}_B\circ f) \qquad \text{for } f\in(A\times B)^{A\times B}.$$

If we consider the space $(A\times B)^{A\times B}$ as a monoid with the multiplication induced from the composition of maps, it is clear that the map

$$\eta:((A\times B)^{A\times B},\circ)\longrightarrow(A^{A\times B}\times B^{A\times B},\otimes)$$

is an isomorphism of topological monoids. Since

$$G(A\times B)\simeq\mathrm{Inv}\,(\pi_0((A\times B)^{A\times B},\mathrm{id}_{A\times B})),$$

we have the following

Lemma 1. *There is the isomorphism of groups*:

$$\eta_*: G(A\times B)\simeq\mathrm{Inv}\,(\pi_0(A^{A\times B}\times B^{A\times B},(\mathrm{pr}_A,\mathrm{pr}_B))).$$

Now we define the multiplication $\tilde{\times} : (A^A\times B^{A\times B})^2\to A^A\times B^{A\times B}$ by

$$(f_1,g_1)\tilde{\times}(f_2,g_2)=(f_1\circ f_2,g_1(f_2\circ\mathrm{pr}_A,g_2)) \qquad \text{for } (f_i,g_i)\in A^A\times B^{A\times B},\quad i=1,2.$$

Then we have a monoid $(A^A\times B^{A\times B},\tilde{\times})$ with the unit $(\mathrm{id}_A,\mathrm{pr}_B)$. In particular, we define the map $\mathrm{res}: A^{A\times B}\to A^A$ by

$$\mathrm{res}\,(f)(a)=f(a,b_0) \qquad \text{for } f\in A^{A\times B} \text{ and } a\in A,$$

where b_0 is the base point of B. Clearly, the map $\mathrm{res}: A^{A\times B}\to A^A$ is a homomorphism of monoids. Here we note the following two lemmas, which can be proved by the standard arguments.

Lemma 2. *The induced homomorphism*

$$\mathrm{pr}_{A*}:\pi_0(A^A,\mathrm{id}_A)\longrightarrow\pi_0(A^{A\times B},\mathrm{pr}_A)$$

is surjective iff $[B,A]=[A\wedge B,A]=\{0\}$.

Lemma 3. *If* $\mathrm{pr}_{A*}:\pi_0(A^A,\mathrm{id}_A)\longrightarrow\pi_0(A^{A\times B},\mathrm{pr}_A)$ *is surjective, then the induced homomorphism*

$$(\mathrm{res}\times\mathrm{id})_*:\pi_0(A^{A\times B},\mathrm{pr}_A)\times\pi_0(B^{A\times B},\mathrm{pr}_B)\longrightarrow\pi_0(A^A,\mathrm{id}_A)\times\pi_0(B^{A\times B},\mathrm{pr}_B)$$

is an isomorphism of monoids.

Then, from the above lemmas we have

Proposition 1. *If* $[B,A]=[A\wedge B,A]=\{0\}$, *the sequence*

$$1\longrightarrow\mathrm{Inv}\,(\pi_0(B^{A\times B},\mathrm{pr}_B))\longrightarrow G(A\times B)\longrightarrow G(A)\longrightarrow 1$$

is split exact.

§ 3. **The proof of Theorem.** First, we define the map $P: B^{A\times B}\to B^B$ by

$$P(g)(b)=g(a_0,b) \qquad \text{for } g\in B^{A\times B} \text{ and } b\in B.$$

Then the map P is a fibration and a homomorphism of monoids. Furthermore, if we define the map $s: B^B \to B^{A\times B}$ by

$$s(f)(a, b) = f(b) \qquad \text{for } f \in B^B \text{ and } (a, b) \in A \times B,$$

then the map s is a cross section of P. Therefore we have

Proposition 2. *The sequence*

$$1 \longrightarrow \mathrm{Inv}\,(\pi_0(P^{-1}(\mathrm{id}_B), \mathrm{pr}_B)) \xrightarrow{i_*} \mathrm{Inv}\,(\pi_0(B^{A\times B}, \mathrm{pr}_B)) \xrightarrow{P_*} G(B) \longrightarrow 1$$

is split exact.

Here we recall the following

Lemma 4. *If B is simply connected, then*

$$\pi_0(P^{-1}(\mathrm{id}_B), \mathrm{pr}_B) \simeq [A, a_0; B^B, \mathrm{id}_B].$$

Secondly we define the maps

$$c_1 : G(A) \longrightarrow G(A \times B),$$
$$c_2 : G(B) \longrightarrow G(A \times B),$$

and

$$c_3 : G(A \times B) \longrightarrow G(B)$$

as follows:

$$c_1(f)(a, b) = (f(a), b) \qquad \text{for } f \in G(A) \text{ and } (a, b) \in A \times B,$$
$$c_2(g)(a, b) = (a, g(b)) \qquad \text{for } g \in G(B) \text{ and } (a, b) \in A \times B,$$

and

$$c_3(h)(b) = \mathrm{pr}_B(h(a_0, b)) \qquad \text{for } h \in G(A \times B) \text{ and } b \in B.$$

Then to prove Theorem, it suffices to show the following lemma, which is obtained by straight-forward calculations.

Lemma 5. (a) $\mathrm{res}_* \circ c_1 = \mathrm{id}_{G(A)}$.

(b) $c_3 \circ c_2 = \mathrm{id}_{G(B)}$.

(c) $c_3 \circ i_* = P_*$.

(d) *The maps c_1 and c_2 are homomorphisms.*

(e) *The map c_3 is a homomorphism if $[B, A] = \{0\}$.*

References

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