89. On a Certain Property of Profinite Groups

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1. We are led to consider a certain property of profinite groups in investigating a problem posed by Jehne [1] on Kronecker sets of algebraic number fields. Let \( \mathbb{Q} \) be the rational number field, \( k \) a finite algebraic extension of \( \mathbb{Q} \) and \( K \) a finite algebraic extension of \( k \). For the extension \( K/k \), we consider the set \( P(K/k) \) of all prime divisors of \( k \) having a prime divisor of first relative degree in \( K/k \). We call this set \( P(K/k) \) the Kronecker set of \( K/k \). For Kronecker sets, we denote equality of sets up to finite set by \( \cong \). Now, the problem of Jehne asks whether there exists a sequence \( \{k_n\}_{n=1}^{\infty} \) of finite algebraic extensions of \( k \) such that \( k_{n+1} \supset k_n \) and that \( P(k_{n+1}) \cong P(k_n/k) \) for any positive integer \( n \). We call the above sequence \( \{k_n\}_{n=1}^{\infty} \) an infinite Kronecker tower of \( k \). In Satz 6 of [2], Klingen claimed that there exists no infinite Kronecker tower of \( k \). As we shall see in the following, the proof of the theorem contains an argument, which is not correct. The following property of Kronecker sets is well-known:

**Proposition 1.** Let \( k \) be a finite algebraic extension of \( \mathbb{Q} \), \( L \) a (finite or infinite) Galois extension of \( k \) and \( G \) the Galois group of \( L \) over \( k \). Let \( H \) and \( H' \) be open subgroups of \( G \). Let \( K \) and \( K' \) be subfields of \( L \) corresponding to the subgroups \( H \) and \( H' \) of \( G \), respectively. Then the following conditions are equivalent:

1. \( P(K/k) \cong P(K'/k) \).
2. \( \bigcup_{g \in G} g^{-1}Hg = \bigcup_{g \in G} g^{-1}H'g \).

We owe the following lemma essentially to Klingen [2]:

**Lemma 1.** Let \( L \) be a (finite or infinite) Galois extension of \( k \) and \( G \) the Galois group \( G(L/k) \) of \( L \) over \( k \). For any positive integer \( n \), we denote by \( k_n \) a finite algebraic extension of \( k \) such that \( L \) contains \( k_n \). We suppose that \( k_{n+1} \) contains \( k_n \) for any positive integer \( n \). Let \( K = \bigcup_{n=1}^{\infty} k_n \), let \( H = G(L/K) \) and let \( H_n = G(L/k_n) \). Then the following conditions are equivalent:

1. \( P(k_{n+1}) \cong P(k_n/k) \) for any positive integer \( n \).
2. \( \bigcup_{g \in G} g^{-1}H_ng = \bigcup_{g \in G} g^{-1}H_ng \).

The following lemma follows immediately from the fact that \( G \) is a profinite group:

**Lemma 2.** Let \( L \) be a (finite or infinite) Galois extension of \( k \) and \( K \) an intermediate field \( L \) over \( k \). Let \( G = G(L/k) \) and \( H = G(L/K) \).
If $\bigcup_{g \in G} g^{-1}Hg$ is an open subset of $G$, then there exists a finite Galois extension $F$ of $k$ such that $L$ contains $F$ and that $\bigcup_{g \in G} g^{-1}(L/F)g = G(L/F)$.

Lemmas 1 and 2 yield the following

Theorem 1 (cf. [2]). Let $k$ be a finite algebraic extension of $Q$, $L$ a (finite or infinite) Galois extension of $k$ and $G$ the Galois group of $L$ over $k$. Then the following conditions are equivalent:

1. The Galois group $G$ has a property that $\bigcup_{g \in G} g^{-1}Hg$ is not open in $G$ for any non-open closed subgroup $H$ of $G$.

2. There exists no infinite Kronecker tower $\{k_n\}_{n=1}^\infty$ of $k$, such that $L$ contains $\bigcup_{n=1}^\infty k_n$.

For convenience's sake, we shall use the following definition:

Definition. A profinite group $G$ is called regular, if for any non-open closed subgroup $H$ of $G$, $\bigcup_{g \in G} g^{-1}Hg$ is not open in $G$.

2. Let $k$ be a finite algebraic extension of $Q$, $\bar{k}$ the algebraic closure of $k$ and $G$, the Galois group of $\bar{k}$ over $k$. In the proof of Satz 6 of [2], it was claimed that $G_\bar{k}$ is regular from the ground that for a non-open closed subgroup $H$ of $G_\bar{k}$, the group index $(G_\bar{k} : H)$ should be countable. This is, however, not the case, as the following shows:

Proposition 2. Let $G$ be a compact group and $H$ a non-open closed subgroup of $G$. Then the group index $(G : H)$ is not countable.

Proof. Let $\mu$ be a Haar measure of $G$ such that $\mu(G) = 1$. Suppose that $(G : H)$ is countable. Let $G = \bigcup_{i=1}^n Hg_i$ be the disjoint union of the right cosets of $H$. We have $\mu(G) = \sum_{i=1}^n \mu(Hg_i)$ and $\mu(Hg_i) = \mu(H)$. If $\mu(H) > 0$, then we have $\mu(G) = \infty$. If $\mu(H) = 0$, then we have $\mu(G) = 0$. This is a contradiction.

3. Now, we shall show that some profinite groups are regular.

Lemma 3. Let $X$ be a topological space, $n$ a positive integer and $X_i$ a non-empty closed subset of $X$ for $i = 1, 2, \ldots, n$. If $\bigcup_{i=1}^n X_i$ is an open subset of $X$, then there exists an integer $i_0 (1 \leq i_0 \leq n)$ such that $X_{i_0}$ contains a non-empty open subset of $X$.

Proof. Let $W = \bigcup_{i=1}^n X_i$, and $m = \min \{l | W = \bigcup_{i=1}^l X_i\}$. Let $W = \bigcup_{i=0}^n X_i$. Then $X_{i_0}$ contains the non-empty open subset $W - \bigcup_{i=1}^l X_i$ of $X$.

Theorem 2. Let $G$ be a profinite group and $N$ a closed abelian normal subgroup of $G$. If the factor group $G/N$ is regular, then $G$ is regular.

Proof. Let $H$ be a closed subgroup of $G$ such that $\bigcup_{g \in G} g^{-1}Hg$ is an open subset of $G$. Since $G/N$ is regular and since the set $\bigcup_{g \in G} (g^{-1}HgN)/N$ is an open subset of $G/N$, the factor group $HN/N$ is an open subgroup of $G/N$. Hence $HN$ is an open subgroup of $G$. Let $G = \bigcup_{i=1}^\infty HNg_i$ be the disjoint union of the right cosets of $HN$. Since $N$ is a normal abelian subgroup of $G$, we have
\[ N \cap \left( \bigcup_{g \in G} g^{-1}Hg \right) = \bigcup_{g \in G} g^{-1}(N \cap H)g = \bigcup_{i=1}^{n} g_{i}^{-1}(N \cap H)g_{i}. \]

Hence, from Lemma 3, we see that \( H \cap N \) is an open subgroup of \( N \). Since the group index \((G:H)\) is equal to \((G:HN)(N:N \cap H)\), \( H \) is an open subgroup of \( G \). Hence we see that \( G \) is regular.

**Theorem 3.** Let \( G \) be a profinite group such that \( G \) contains an open normal solvable subgroup of \( G \). Then \( G \) is regular.

**Proof.** There exists a sequence \( \{ N_{i} \}_{i=0}^{r} \) of normal closed subgroups of \( G \), which satisfies the following conditions:

1. \( N_{0} \supseteq N_{1} \supseteq \cdots \supseteq N_{r} \).
2. \( N_{i} \) is an open normal subgroup of \( G \).
3. \( N_{i}/N_{i+1} \) is an abelian group for \( i = 0, 1, \ldots, r-1 \).
4. \( N_{r} = \{1\} \).

We use induction on \( r \). If \( r = 0 \), then \( G \) is a finite group, which shows that \( G \) is regular. If \( r > 0 \), by applying the induction, we see that \( G/N_{r-1} \) is regular. Then, from Theorem 2, we see that \( G \) is regular.

4. Now we shall show the following

**Theorem 4.** A pro-nilpotent group is regular.

For the proof we need the following

**Lemma 4.** Let \( G \) be a nilpotent group, \( N \) a normal subgroup of \( G \) and \( H \) a subgroup of \( N \). If \( N = \bigcup_{g \in G} g^{-1}Hg \), then we have \( H = N \).

**Proof.** There exists a sequence \( \{ N_{i} \}_{i=0}^{r} \) of normal subgroups of \( G \) satisfying the following conditions:

1. \( N_{0} \supseteq N_{1} \supseteq \cdots \supseteq N_{r} = \{1\} \).
2. \( N_{0} = N \).
3. \( N_{i}/N_{i+1} \) is contained in the center of \( G/N_{i+1} \) for \( i = 0, 1, \ldots, r-1 \). We use induction on \( r \). If \( r = 0 \), it is clear that \( H = N \). If \( r > 0 \), by applying the induction assumption, we have \( HN_{r-1} = N \). Since we have

\[ N_{r-1} = \left( \bigcup_{g \in G} g^{-1}Hg \right) \cap N_{r-1} = \bigcup_{g \in G} g^{-1}(H \cap N_{r-1})g = H \cap N_{r-1}, \]

we have \( H = N \).

**Proof of Theorem 4.** Let \( G \) be a pro-nilpotent group and \( H \) a closed subgroup of \( G \) such that \( \bigcup_{g \in G} g^{-1}Hg \) is an open subset of \( G \). Then there exists an open normal subgroup \( N \) of \( G \) such that \( \bigcup_{g \in G} g^{-1}Hg \supseteq N \). Let \( H_{i} = N \cap H \). We have \( N = \bigcup_{g \in G} g^{-1}H_{i}g \). Let \( N_{*} \) be any open normal subgroup of \( G \) such that \( N \supseteq N_{*} \). Then we have \( N/N_{*} = \bigcup_{g \in G} (g^{-1}H_{i}gN_{*})/N_{*} \), which shows \( N = H_{i}N_{*} \) from Lemma 4. Hence we have \( N = H_{i} \).

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References
