86. A Characterization of Hyperplane Cuts of Smooth Complete Intersections

By Shihoko ISHII

Department of Mathematics, Tokyo Metropolitan University

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In this note, we will prove the following

Theorem. *Let $M\subset \boldsymbol{P}^{N+1}$ be a smooth complete intersection. We assume for simplicity that M is non-degenerate,* i.e., *M is not contained in any linear subspace of $\boldsymbol{P}^{N+1}$. Then any hyperplane section X of M has the following two properties:*

(A) *X has only finitely many singular points;*

(B) *The Jacobian matrix J of $X\subset \boldsymbol{P}^N$ has rank $r-1$ at any singular point of X.*

Conversely, if $X\subset \boldsymbol{P}^N$ is a non-degenerate complete intersection having the properties (A) *and* (B), *then there exists a smooth complete intersection $M\subset \boldsymbol{P}^{N+1}$ such that X is a hyperplane section of M.*

Remark. The property (A) implies that X is reduced if $\dim M\geq 2$ and irreducible if $\dim M\geq 3$. Moreover, (A) is a partial refinement of the following

Zak's Theorem (see [1]). *Let $M\subset \boldsymbol{P}^{N+1}$ be an irreducible smooth non-degenerate subvariety of codimension r and X an arbitrary hyperplane section of M. Then the dimension of the singular locus of X is less than r.*

In [1], the property (A) is shown by using a suitable incidence correspondence. Our proof is more direct and elementary.

Throughout this note, we fix an algebraically closed field k of any characteristic and assume that all varieties are defined over k.

Proof of Theorem. For brevity, we introduce a symbol $V(F_1,\cdots,F_r)$ which stands for the projective variety defined by the homogeneous polynomials $F_1,\cdots,F_r$. For a given smooth complete intersection $M\subset \boldsymbol{P}^{N+1}$, we write $M=V(\tilde{F}_1,\cdots,\tilde{F}_r)$, where $\tilde{F}_i$ is a homogeneous polynomial of degree $d_i\geq 2$ in $Z_0,Z_1,\cdots,Z_{N+1}$. By a suitable linear transformation of the coordinates, we may assume that

$$X=M\cap\{Z_{N+1}=0\}=V(\tilde{F}_1,\cdots,\tilde{F}_r,Z_{N+1}).$$

Putting $F_i(Z_0,\cdots,Z_N)=\tilde{F}_i(Z_0,\cdots,Z_N,0)$, we write

$$\tilde{F}_i(Z_0,\cdots,Z_{N+1})=F_i(Z_0,\cdots,Z_N)+Z_{N+1}G_i(Z_0,\cdots,Z_{N+1}).$$

Denote by $\tilde{J}(p)$ and $J(p)$ the Jacobian matrices of the defining equations $\{\tilde{F}_1,\cdots,\tilde{F}_r\}$ and $\{F_1,\cdots,F_r\}$ at $p\in X$, respectively. Then, since $Z_{N+1}=0$ on X, we have

$$\tilde{J}(p)=\begin{pmatrix} J(p) \\ G_1(p), \cdots, G_r(p) \end{pmatrix}.$$

Since M is smooth, rank $\tilde{J}(p)=r$. So we have rank $J(p)\geq r-1$. This implies (B).

Let S be an irreducible component of the singular locus of X. We assume that S is of maximal dimension. Noting that rank $J(p)=r-1$ at $p\in S\subset X$, we choose a non-zero vector $(a_1(p), \cdots, a_r(p))$ for each $p\in S$ such that

$$\sum_{i=1}^{r} a_i(p)\frac{\partial F_i}{\partial Z_j}(p)=0 \qquad (j=0, \cdots, N).$$

Then the vector $f(p)=(a_1(p)G_1(p), \cdots, a_r(p)G_r(p))$ determines a point in $\boldsymbol{P}^{r-1}$, which does not depend on the vector $(a_1(p), \cdots, a_r(p))$. In fact, if $a_1(p)G_1(p)=\cdots=a_r(p)G_r(p)=0$, then $\sum a_i(p)(\partial\tilde{F}_i/\partial Z_j)(p)=0$ and $\tilde{J}(p)$ would have rank $\leq r-1$. Thus, $f: S\rightarrow \boldsymbol{P}^{r-1}$ is a morphism. Assume that f is not a constant map. Then $f(S)\cap\{Y_1+\cdots+Y_r=0\}\neq\phi$, where $\{Y_1, \cdots, Y_r\}$ are the homogeneous coordinates of $\boldsymbol{P}^{r-1}$. This implies that

$$\sum a_i(p)G_r(p)=\sum a_i(p)\frac{\partial F_i}{\partial Z_j}(p)=0 \qquad (j=0, 1, \cdots, N)$$

for some point $p\in S$ and M would not be smooth at p. Hence f must be a constant map, and $a_i(p)G_i(p)$ never vanish on S for some i. This is possible only when $\dim S=0$ since $\deg G_i=d_i-1\geq 1$.

Now, let X be a complete intersection $V(F_1, \cdots, F_r)$ in $\boldsymbol{P}^N$ ($d_i=\deg F_i\geq 2$), and assume that X has the above properties (A) and (B). Let S be the finite singular locus of X and let $G_i^{(k)}(Z_0, \cdots, Z_N)$ be a general homogeneous polynomial of degree d_i-k. Set $\tilde{F}_i(Z_0, \cdots, Z_{N+1})=F_i(Z_0, \cdots, Z_N)+\sum_{k=1}^{d_i} Z_{N+1}^k G_i^{(k)}(Z_0, \cdots, Z_N)$. Then the Jacobian matrix $\tilde{J}(p)$ of $M=V(\tilde{F}_1, \cdots, \tilde{F}_r)\subset\boldsymbol{P}^{N+1}$ at $p\in X$ is $\begin{pmatrix} J(p) \\ G_1^{(1)}(p), \cdots, G_r^{(1)}(p)\end{pmatrix}$. Since S is finite and $G_i^{(1)}$ is general, rank $\tilde{J}(p)$ is equal to r at $p\in S$. Hence $\tilde{J}$ has rank r everywhere on $X=M\cap V(Z_{N+1})$. On the other hand, $M\backslash X$ is non-singular in virtue of the following

Lemma. *Let $f_1, \cdots, f_r$ be polynomials in $x_1, \cdots, x_n$ $(1\leq r\leq n)$. Then the affine variety in A^n defined by*

$$f_i+\sum_{j=1}^{n} a_{ij}x_j+a_{in+1} \qquad (i=1, \cdots, r)$$

is non-singular for general coefficients $(a_{ij})\in k^{r(n+1)}$.

Remark. The above theorem fails for a general smooth submanifold $M\subset\boldsymbol{P}^{N+1}$. Indeed, a hypersurface section of M satisfies neither (A) nor (B) in general. Therefore, the Veronese embedding of M gives a counterexample.

Reference

[1] W. Fulton and R. Lazarsfeld: Connectivity and its applications in algebraic geometry. Lect. Note in Math., vol. 862, Springer-Verlag, pp. 26–92 (1980).