85. On the Generic Existence of Holomorphic Sections and Complex Analytic Bordism

By Hiroshi MORIMOTO
Nagoya University

(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 13, 1982)

§0. **Introduction.** In this paper, we discuss the generic existence of certain holomorphic sections of holomorphic vector bundles over compact complex manifolds. Moreover we study the relation among Schubert cycles associated with holomorphic sections from a view point of complex analytic bordism.

Let M be a compact complex manifold and $E \to M$ be a holomorphic vector bundle of rank q. For holomorphic sections $\sigma_1, \cdots, \sigma_r$ of $E \to M$, there is an associated Schubert cycle which is defined to be the subset of M where $\sigma_1, \cdots, \sigma_r$ are not linearly independent. Holomorphic sections $\sigma_1, \cdots, \sigma_r$ are said to be in general position if the associated Schubert cycle has codimension $q-r+1$. The bundle $E \to M$ is said to be sufficiently ample if there are global sections $\tau_1, \cdots, \tau_N \in \mathcal{O}(M, E)$ such that (i) the $\tau_i(x)$ span all fibres E_x $(x \in M)$ and that (ii) the differentials $d\tau(x)$ of the sections $\tau = \sum a_i \tau_i (a_i \in C)$ which vanish at x span $E_x \otimes \mathcal{T}_x^*$. Cornalba and Griffiths stated in [1] that if the bundle is sufficiently ample, then suitable linear combinations $\sigma_v = \sum a_{vi} \tau_i$ $(v=1, \cdots, r)$ give sections $\sigma_1, \cdots, \sigma_r$ which are in general position. The usefulness of the general position requirement lies in the fact that in this case the associated Schubert cycle represents the Chern class of the given bundle.

One of our results is to show a stronger result by omitting the condition (ii). Moreover it will be shown only under the condition (i) that one can deform arbitrary sections $\sigma_1, \cdots, \sigma_r$ so that the associated Schubert cycle has only singularities of simple type which we call *quasilinear* type. In this case, the deformed sections $\tilde{\sigma}_1, \cdots, \tilde{\sigma}_r$ are said to be in *quasilinear* position. Our generic existence Theorem 3.1 asserts the openness and density of holomorphic sections in quasilinear position.

If two sets of holomorphic sections $\sigma_1, \cdots, \sigma_r$ and $s_1, \cdots, s_r$ are in quasilinear position, then the associated Schubert cycles are homologous, because they represent the same cohomology class. In this paper, we shall show that they are more intimately related. In fact, our bordism theorem shows that there exists a complex analytic bordism of quasilinear type which connect them. The detailed proof will be given elsewhere.

§1. Definition of quasilinear structure. A quasilinear subvariety $V \subset M$ is characterized by its structure of stratification. The quasilinear structure is modeled on some corns in the space of complex matrices. We shall denote by $\mathfrak{M}(r,s)$ the set of all $r \times s$ complex matrices. We set

$$\mathfrak{M}_k(r,s) = \{A \in \mathfrak{M}(r,s)\,;\ \text{corank}\,(A) \geq k\},$$

where $r \leq s$, $k = 1, \cdots, r$. Then we have a sequence of subvarieties in $\mathfrak{M}(r,s) \times \boldsymbol{C}^t$ for any integer t,

$$\mathfrak{M}_1(r,s) \times \boldsymbol{C}^t \supset \mathfrak{M}_2(r,s) \times \boldsymbol{C}^t \supset \cdots \supset \mathfrak{M}_r(r,s) \times \boldsymbol{C}^t = 0_{r,s} \times \boldsymbol{C}^t,$$

where $0_{r,s}$ is the zero matrix. We regard the submanifold $0_{r,s} \times \boldsymbol{C}^t$ as a center. Our model is the structure of stratification in the neighbourhood of this center.

Definition 1.1. A complex analytic subvariety $V \subset M$ is said to be *quasilinear* if it has a regular stratification $V = V_1 \supset \cdots \supset V_N$ such that for any point z in any stratum $V_i - V_{i+1}$ there exist some integer r, s, t and a biholomorphic map φ of a neighbourhood U of z onto a neighbourhood W of $(0_{r,s}, 0)$ in $\mathfrak{M}(r,s) \times \boldsymbol{C}^t$ such that

$$\varphi(U \cap V_k) = W \cap (\mathfrak{M}_k(r,s) \times \boldsymbol{C}^t), \qquad k = 1, \cdots, i.$$

We also say that V has singularities of quasilinear type.

§2. Schubert cycle and quasilinear position. Given holomorphic sections $\sigma_1, \cdots, \sigma_r$ of holomorphic vector bundle $E \to M$, we define the associated Schubert cycle $\mathcal{S} = \mathcal{S}(\sigma_1, \cdots, \sigma_r)$ by setting

$$\mathcal{S}(\sigma_1, \cdots, \sigma_r) = \{z \in M\,;\ \sigma_1(z) \wedge \cdots \wedge \sigma_r(z) = 0\}.$$

Let $p \in \mathcal{S}$ and let $\sigma_1, \cdots, \sigma_r$ be represented as $\sigma_i(x) = \sum \alpha_{ij}(z) e_j(z)$ for some local holomorphic frame $e_1, \cdots, e_q$, $q = \text{rank}\,(E)$ on a neighbourhood U at p in M. Then $\alpha_{ij}(z)$ define holomorphic map $\Phi(z) = (\alpha_{ij}(z))$ of U into $\mathfrak{M}(r,q)$. Notice that $\Phi(p) \in \mathfrak{M}_1(r,q)$ because $p \in \mathcal{S}$.

Definition 2.1. Holomorphic sections $\sigma_1, \cdots, \sigma_r$ are said to be in *quasilinear* position if at any point $p \in \mathcal{S}(\sigma_1, \cdots, \sigma_r)$, the map Φ is transversal to any stratum of $\mathfrak{M}_1(r,q)$ at any point of U.

Quasilinear position has the following meaning.

Lemma 2.2. *If holomorphic sections $\sigma_1, \cdots, \sigma_r$ are in quasilinear position, then the associated Schubert cycle $\mathcal{S}(\sigma_1, \cdots, \sigma_r)$ has only singularities of quasilinear type.*

§3. Generic existence theorem. In this section we are concerned with the openness and the density of holomorphic sections in quasilinear position. In case M is a compact complex manifold, the topology of the set of all the family of holomorphic r-sections $\sigma_1, \cdots, \sigma_r$ which will be denoted by $\oplus^r \Gamma(M, \mathcal{O}(E))$ is naturally defined including higher order differentials. Our generic existence theorem is stated as follows.

Theorem 3.1. *Let M be a compact complex manifold and $E \to M$ be a holomorphic vector bundle which satisfies the condition* (i) *in the introduction. Then, the set of holomorphic r-sections $\{\sigma_1, \cdots, \sigma_r\}$ which are in quasilinear position is open and dense in the space $\oplus^r \Gamma(M, \mathcal{O}(E))$.*

§ 4. Quasilinear bordism theorem. In this section we define quasilinear bordism for quasilinear subvarieties in M and we show the existence of such bordism between Schubert cycles of holomorphic sections in quasilinear position.

Definition 4.1. Quasilinear subvarieties of dimension k, X_1 and X_2 in M are said to be *quasilinearly bordant* in the strong sense if there is a quasilinear subvariety W of dimension $k+1$ in $M \times \boldsymbol{C}$ such that $X_1 = W \cap (M \times \{0\})$ and $X_2 = W \cap (M \times \{1\})$.

Definition 4.2. Quasilinear subvarieties V_1 and V_2 in M are said to be *quasilinearly bordant* if there is a sequence of quasilinear subvarieties $X_1, \cdots, X_N$ in M such that X_i and X_{i+1} are quasilinearly bordant in the strong sense for any $0 \leq i \leq N+1$, where we set $V_1 = X_0$, $V_2 = X_{N+1}$.

Now we are in a position to state our bordism theorem.

Theorem 4.3. *Let M be a compact complex manifold and $E \to M$ be a holomorphic vector bundle which satisfies the condition* (i) *in the introduction. If holomorphic sections $\sigma_1, \cdots, \sigma_r$ and $s_1, \cdots, s_r$ are both in quasilinear position, then, their associated Schubert cycles $\mathcal{S}(\sigma_1, \cdots \sigma_r)$ and $\mathcal{S}(s_1, \cdots, s_r)$ are quasilinearly bordant in M.*

References

[1] M. Cornalba and P. Griffiths: Analytic cycles and vector bundles on non-compact algebraic varieties. Invent. math., **28**, 1–106 (1975).

[2] H. Morimoto: Realization of Chern classes by subvarieties with certain singularities. Nagoya Math. J., **80**, 49–74 (1980).

[3] ——: On certain complex analytic cobordism between subvarieties realizing Chern classes of bundles. ibid., **88** (1982) (to appear).