83. *On 4-Manifolds Fibered by Tori*

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§1. **Definitions.** The class of elliptic surfaces plays an important role in the theory of complex surfaces, [2]. In this note, we consider an analogous structure on smooth 4-manifolds, which we call *torus fibration*, and announce some results. Before giving the definition of torus fibration, we need slightly extend the notion of fibered link in the 3-sphere.

Definition. A smooth map $g: S^3 \to \boldsymbol{C}$ is called a *multiple fibered link* if it satisfies the following:

(i) $g^{-1}(0) \neq \phi$;

(ii) the map $\varphi(\boldsymbol{x}) = g(\boldsymbol{x})/|g(\boldsymbol{x})| : S^3 - g^{-1}(0) \to S^1$ is a submersion;

(iii) at each $\boldsymbol{x}_0 \in g^{-1}(0)$, there exist local coordinates u_1, u_2, u_3 in S^3 so that

$$g(\boldsymbol{x}) = (u_2(\boldsymbol{x}) + \sqrt{-1}u_3(\boldsymbol{x}))^m$$

holds for all $\boldsymbol{x}$ near $\boldsymbol{x}_0$, m being a certain positive integer (called the *multiplicity at* $\boldsymbol{x}_0$).

Definition. A map $f: \boldsymbol{R}^4 \to \boldsymbol{C}$ is a *cone-extension* of a smooth map $g: S^3 \to \boldsymbol{C}$, if it is given as follows:

$$f(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|^d g(\boldsymbol{x}/\|\boldsymbol{x}\|) & \boldsymbol{x} \neq \boldsymbol{0} \\ 0 & \boldsymbol{x} = \boldsymbol{0} \end{cases}$$

where d is an integer >0 depending on f.

Clearly f is smooth outside of the origin $\boldsymbol{0} \in \boldsymbol{R}^4$. Let $f_i: M_i^m \to N_i^k$ be a map, $p_i \in M_i^m$ a point, for $i=1,2$, where M_i^m and N_i^k are oriented smooth manifolds. We say that the germ (f_1, p_1) is *smoothly* $(+)$-*equivalent* to the germ (f_2, p_2) if they are equivalent through orientation preserving local diffeomorphisms around p_i and $f_i(p_i)$.

Now we define the torus fibration. Let M and B be oriented smooth manifolds of dimensions 4 and 2, respectively. In this note, we assume that M and B are closed for the sake of convenience.

Definition. *A torus fibration of M with base space B* is an onto map $f: M \to B$ with the following properties:

(i) at each point $p \in M$, the germ (f, p) is smoothly $(+)$-equivalent to the germ at $\boldsymbol{0}$ of a cone-extension of a multiple fibered link;

(ii) the inverse image $C_u = f^{-1}(u)$ of any *general* point $u \in B$ is diffeomorphic to the 2-torus T^2.

Note that the projection map f is smooth outside a finite set of points. A special type of torus fibration has been studied by Moishezon [4] and Harer [7].

Torus fibration is an underlying structure of elliptic surfaces:

Proposition 1. *Suppose that $\Phi: M\to B$ is an analytic fiber space of elliptic curves* [2], *then there exist a torus fibration $f: M\to B$ and an orientation preserving homeomorphism $h: M\to M$ which is a diffeomorphism outside a finite set of points, so that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "\Phi"'] & & M \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

Given a torus fibration, we can define (multiple or simple) singular fibers, devisors, the monodromy around a singular fiber, etc. in the same way as in the case of complex surfaces.

§2. An existence theorem. There are several necessary conditions for a 4-manifold to admit a torus fibration.

Proposition 2. *If $f: M\to B$ is a torus fibration, then the Euler number $\chi(M)$ is non-negative.*

It is shown that the fundamental group of a singular fiber is either $\{1\}$, $\boldsymbol{Z}$ or $\boldsymbol{Z}\oplus\boldsymbol{Z}$. Using this fact, Koichi Yano proved the following theorem (he remarks that this theorem also follows from a result of Gromov [1, §3.1]):

Theorem 3 (Yano, Gromov). *If $f: M\to B$ is a torus fibration, then the Gromov invariant of M vanishes.*

We have the following existence theorem:

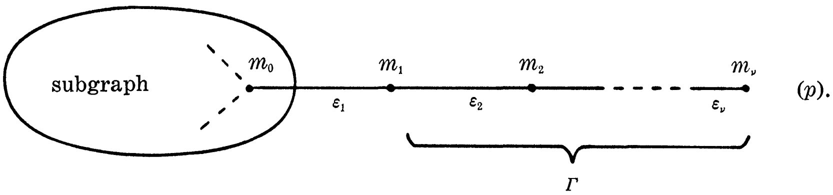
Theorem 4. *Suppose that M has a handle-body decomposition of the form $M=H^0\cup\mu H^2\cup\nu H^3\cup H^4$ ($\nu\leqq 1$). Then there exists a torus fibration $f: M\to S^2$.*

According to Mandelbaum [3], every nonsingular complete intersection of k distinct hypersurfaces in the complex projective space $\boldsymbol{CP}^{k+2}$ has a handle-body decomposition without 1 and 3 handles. Thus it admits a torus fibration.

§3. A torus fibration of S^4. The 4-sphere S^4 has a torus fibration. The author's original construction was inspired by Montesinos' work [6] and made use of the Heegaad diagrams of 4-manifolds [5]. Afterwards, K. Fukaya gave the following nice construction: Let $H: S^3\to S^2$ be the Hopf fibration, $\Sigma H: S^4\to S^3$ its suspension. Then the composition $H\circ\Sigma H: S^4\to S^2$ is a torus fibration. If S^3 is identified with ΣS^2 so that a fiber of H goes through the two suspension vertices, then the singular fiber consists of two S^2's which intersect transversely in two points with opposite signs (a *twin* in the sense of [6]). As we change the identification of S^3 with ΣS^2, the torus fibration deforms, and the twin singular fiber splits into two simpler singular fibers.

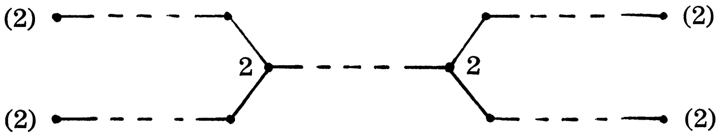
§ 4. **Singular fibers with normal crossings.** To obtain somewhat deeper results, certain restrictions on the type of singular fibers would be desirable. A singular fiber is said to be of *normal type*, if it consists of smoothly embedded S^2's or T^2 which intersect transversely. Such a fiber is represented by a dual weighted graph, in which each vertex stands for an embedded 2-sphere. Two vertices are joined by an edge (labelled with sign $\varepsilon=\pm 1$) if and only if the corresponding 2-spheres transversely intersect in a point with the sign ε. The weight of a vertex represents the multiplicity.

Consider the linear branch Γ

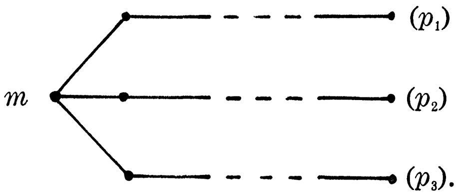


It is shown that $\gcd(m_0, m_1)=\gcd(m_1, m_2)=\cdots=\gcd(m_{\nu-1}, m_\nu)=m_\nu$. The number $p=p(\Gamma)$ in the parentheses is given by $p=m_0/m_\nu$. A *removable linear branch* (RLB) is a linear branch Γ for which $p=1$ holds. The neighbourhood boundary of an RLB is S^3, and if we 'remove' an RLB from a singular fiber, the monodromy remains unaffected.

Theorem 5. *Singular fibers of normal type without RLB are classified into the following six classes:* (i) *class mI_0 (multiple tori),* (ii) *class $\tilde{A}$ in which the graphs are cyclic* (iii) *class $\tilde{D}$ in which the graphs are of the form*



(iv) *class $\tilde{E}_6$ with $(m=3, p_1=p_2=p_3=3)$,* (v) *class $\tilde{E}_7$ with $(m=4, p_1=2, p_2=p_3=4)$,* (vi) *class $\tilde{E}_8$ with $(m=6, p_1=2, p_2=3, p_3=6)$, where in the last three classes the graphs have the form*



When all the signs of intersections are $+1$, these singular fibers reduce to Kodaira's singular fibers [2] or their blown ups. For example, the class $\tilde{E}_8$ reduces to {II, II*}.

The detailed proofs will appear elsewhere.

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