80. A Note on Modularity in Atomistic Lattices

By Shûichirô MAEDA
Department of Mathematics, Ehime University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 13, 1982)

Let L be an atomistic lattice ([1], (7.1)), and let A, B be subsets of L. If (a, b) is a modular pair (resp. dual-modular pair) for every $a \in A$ and $b \in B$, we write $(A, B)M$ (resp. $(A, B)M^*$). We denote by Ω the set of atoms of L, and we put

$$\Omega^n = \{p_1 \vee \cdots \vee p_n ; p_i \in \Omega\} \qquad (n=1, 2, \cdots).$$

Evidently, $\Omega^1 = \Omega$ and $\Omega^n \subset \Omega^{n+1}$. Moreover, we put

$$F = \bigcup_{n=1}^{\infty} \Omega^n \cup \{0\}.$$

$(L, F)M$ means that L is finite-modular ([1], (9.1)), and each of $(\Omega, L)M$ and $(\Omega, L)M^*$ is equivalent to that L has the covering property ([1], (7.6)). If $A_1 \subset A_2$ and $B_1 \subset B_2$, then evidently $(A_2, B_2)M$ implies $(A_1, B_1)M$, and $(A_2, B_2)M^*$ implies $(A_1, B_1)M^*$.

In the previous paper [3], the following equivalences and non-trivial implications were proved:

(1) For any $A \subset L$, $(A, L)M \Longleftrightarrow (A, L)M^*$, $(A, F)M \Longleftrightarrow (A, F)M^*$, $(A, \Omega^n)M \Longleftrightarrow (A, \Omega^{n-1})M^*$ $(n \geq 2)$. ($(L, \Omega)M$ always holds.)

(2) $(L, F)M^* \Longrightarrow (F, L)M^*$.

(3) $(L, \Omega^n)M^* \Longleftrightarrow (L, F)M^*$ for $n \geq 1$.

(4) $(F, \Omega^n)M^* \Longleftrightarrow (F, F)M^*$ for $n \geq 1$.

(5) $(\Omega^n, F)M^* \Longleftrightarrow (F, F)M^*$ for $n \geq 2$.

(6) $(\Omega^n, \Omega)M^* \Longleftrightarrow (\Omega^{n-1}, \Omega^2)M^* \Longleftrightarrow \cdots \Longleftrightarrow (\Omega^2, \Omega^{n-1})M^*$ for $n \geq 3$.

(7) $(\Omega^2, \Omega^{n-1})M^* \Longrightarrow (\Omega, \Omega^n)M^*$ for $n \geq 2$.

Moreover, it was shown by examples that the implications (2) and (7) and the following implications are not reversible:

$$(\Omega^2, L)M^* \Longrightarrow (\Omega^2, F)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^2, \Omega^n)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^2, \Omega)M^*,$$
$$(\Omega, L)M^* \Longrightarrow (\Omega, F)M^* \Longrightarrow \cdots \Longrightarrow (\Omega, \Omega^n)M^* \Longrightarrow \cdots \Longrightarrow (\Omega, \Omega)M^*,$$
$$(\Omega^2, L)M^* \Longrightarrow (\Omega, L)M^*, \qquad (\Omega^2, F)M^* \Longrightarrow (\Omega, F)M^*.$$

But, it remained open whether the following implications are reversible or not:

$$(F, L)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^n, L)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^2, L)M^*.$$

In this paper, we shall prove that these implications are reversible, that is,

Theorem. *For an atomistic lattice L,*

(8) $(\Omega^n, L)M^* \Longleftrightarrow (F, L)M^*$ *for* $n \geq 2$.

To prove this theorem, we prepare the following lemma which follows from [1], (1.5) by the duality.

Lemma. *Let a, b and c be elements of a lattice L.*

(i) *If $(a, b)M^*$ and $(a\vee b, c)M^*$ then $(a_1, b\vee c)M^*$ for any $a_1 \in L[a, a\vee c]$.*

(ii) *If $(a, b)M^*$ then $(a_1, b_1)M^*$ for any $a_1 \in L[a, a\vee b]$ and $b_1 \in L[b, a\vee b]$.*

Proof of the theorem. It suffices to prove that $(\Omega^n, L)M^*$ implies $(\Omega^{n+1}, L)M^*$ for $n\geq 2$. Assume $(\Omega^n, L)M^*$, and let $u\in\Omega^{n+1}$, $a\in L$. We put $u=p_0\vee p_1\vee\cdots\vee p_n$ where $p_i\in\Omega$. If $p_i\leq a\vee p_0\vee p_1\vee\cdots\vee p_{i-1}$ for some i $(0\leq i\leq n)$, then putting $v=p_0\vee p_1\vee\cdots\vee p_{i-1}\vee p_{i+1}\vee\cdots\vee p_n$, we have $v\in\Omega^n$ and $a\vee v=a\vee u$. Since $(v, a)M^*$ by the assumption and since $u\in L[v, v\vee a]$, we have $(u, a)M^*$ by (ii) of the above lemma. Hence, we may assume that

$$(*)\qquad p_i\not\leq a\vee p_0\vee p_1\vee\cdots\vee p_{i-1}\qquad\text{for every } i=0, 1, \cdots, n.$$

Since $(\Omega^n, L)M^*$ implies the covering property, L is an AC-lattice ([1], (8.7)) and hence $L[a, a\vee u]$ is also an AC-lattice by [1], (8.18). Hence, for every $x\in L[a, a\vee u]$ we can define the height $h(x)$ of x in $L[a, a\vee u]$ ([1], (8.5)). It follows from $(*)$ that $h(a\vee u)=n+1$. Now, we shall show that

$$(**)\qquad (c\wedge u)\vee a=c$$

for every $c\in L[a, a\vee u]$. First, we assume $h(c)\leq n-1$. We put $v=p_1\vee\cdots\vee p_n$ and $v'=(p_0\vee c)\wedge v$. If $p_0\vee c\geq v$, then we would have $p_0\vee c\geq p_0\vee v\vee a=a\vee u$ and then $n+1=h(a\vee u)\leq h(p_0\vee c)\leq h(c)+1\leq n$, a contradiction. Hence, $p_0\vee c\not\geq v$ and hence $v'<v$. We have $v'\in\Omega^{n-1}$ since $v\in\Omega^n$, and hence $p_0\vee v'\in\Omega^n$. Using $(p_0\vee v', a)M^*$ and $(v, p_0\vee a)M^*$, we obtain

$$\begin{aligned}(c\wedge u)\vee a&=(c\wedge(p_0\vee v))\vee a\geq(c\wedge(p_0\vee v'))\vee a=c\wedge(p_0\vee v'\vee a)\\&=c\wedge(((p_0\vee c)\wedge v)\vee p_0\vee a)=c\wedge(p_0\vee c)\wedge(v\vee p_0\vee a)\\&=c\wedge(u\vee a)=c\geq(c\wedge u)\vee a,\end{aligned}$$

which implies $(**)$. Next, if $h(c)=n$, then there exist $c_1, c_2\in L[a, a\vee u]$ such that $h(c_1)=n-1$, $h(c_2)=1$ and $c=c_1\vee c_2$. Since $n-1\geq 1$, $(c_i\wedge u)\vee a=c_i$ $(i=1, 2)$ as above. Hence,

$$(c\wedge u)\vee a\geq(c_1\wedge u)\vee(c_2\wedge u)\vee a=c_1\vee c_2=c\geq(c\wedge u)\vee a.$$

If $h(c)=n+1$, then $(**)$ holds since $c=a\vee u$.

If $d\geq a$, then putting $c=d\wedge(a\vee u)$, we have $c\in L[a, a\vee u]$ and $c\wedge u=d\wedge u$. Hence, by $(**)$ we have

$$(d\wedge u)\vee a=(c\wedge u)\vee a=c=d\wedge(u\vee a).$$

Therefore $(u, a)M^*$ holds.

Remark. In [3], the six statements (2)–(7) were proved by the aid of the concept of P-relation, introduced in [2]. We remark that three of them directly follow from (i) of the above lemma. We can show the following statement:

(9) *For any* $A \subset L$, $(A \vee \Omega_0, \Omega^{n-1})M^* \Longrightarrow (A, \Omega^n)M^*$ $(n \geq 2)$, *where* $A \vee \Omega_0 = \{a \vee p\,;\, a \in A,\, p \in \Omega \cup \{0\}\}$.

In fact, if $a \in A$ and $u \in \Omega^n$, then putting $u = p \vee v$ with $p \in \Omega$ and $v \in \Omega^{n-1}$, we have $(a, p)M^*$ and $(a \vee p, v)M^*$ by $(A \vee \Omega_0, \Omega^{n-1})M^*$, and hence $(a, p \vee v)M^*$ by the lemma.

Now, it is easy to verify that (3) and (4) follows from (9), since if $A = L$ or F then $A \vee \Omega_0 = A$. Moreover, it follows from (9) that

$$(\Omega^n, \Omega)M^* \Longrightarrow (\Omega^{n-1}, \Omega^2)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^2, \Omega^{n-1})M^* \Longrightarrow (\Omega, \Omega^n)M^*,$$

which includes (7) and a half of (6).

References

[1] F. Maeda and S. Maeda: Theory of Symmetric Lattices. Springer-Verlag, Berlin (1970).

[2] S. Maeda: On finite-modular atomistic lattices. Algebra Universalis, **12**, 76–80 (1981).

[3] ——: On modularity in atomistic lattices (to appear in Colloq. Math. Soc. János Bolyai, **33** (Contributions to lattice theory)).