79. *Meromorphic Solutions of Some Difference Equations of Higher Order. II*

By Niro YANAGIHARA
Department of Mathematics, Chiba University

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1. Introduction. In this note, we will study the difference equation of order n:

(1.1) $\alpha_n y(x+n)+\alpha_{n-1}y(x+n-1)+\cdots+\alpha_1 y(x+1)=R(y(x))$,

where $R(w)$ is a rational function of w:

(1.2) $$\begin{cases} R(w)=P(w)/Q(w), \\ P(w)=a_p w^p+\cdots+a_1 w+a_0, \\ Q(w)=b_q w^q+\cdots+b_1 w+b_0, \end{cases}$$

in which $\alpha_n, \cdots, \alpha_1$; $a_p, \cdots, a_0$; $b_q, \cdots, b_0$ are consts, and $\alpha_n a_p b_q \neq 0$. $P(w)$ and $Q(w)$ are supposed to be mutually prime. In the below, we denote by p and q the degrees of the nominator $P(w)$ and of the denominator $Q(w)$, respectively. We put

(1.3) $$q_0=\max(p, q).$$

When $n=1$, the equation (1.1) reduces to

(1.4) $$y(x+1)=R(y(x)).$$

Some properties of meromorphic solutions of (1.4) are studied in [1]–[3]. Especially, we proved in [2, p. 311, Theorem 1], that

(1.5) $\begin{cases} \text{any meromorphic solution of (1.4) is transcendental and} \\ \text{of order } \infty \text{ in the sense of Nevanlinna, if } q_0 \geqq 2. \end{cases}$

(1.5) is not valid if $n>1$, but we proved in [4],

Proposition 1. *When $p>q$, then any meromorphic solution of* (1.1) *is transcendental.*

Proposition 2. *When $p>q+1$, then any meromorphic solution of* (1.1) *is of order ∞ in the sense of Nevanlinna.*

Proposition 3. *When $q_0>n$, then any meromorphic solution of* (1.1) *is transcendental and of order ∞ in the sense of Nevanlinna.*

We will show that Propositions 1–3 are exact, i.e.,

Theorem 1. *Suppose $p \leqq q \leqq n$. Then there is an equation of the form* (1.1) *which admits a rational solution.*

Theorem 2. *Suppose $p=q+1 \leqq n$. Then there is an equation of the form* (1.1) *which admits a transcendental solution of finite order.*

Theorem 3. *Suppose $p \leqq q \leqq n$. Then there is an equation of the form* (1.1) *which admits a transcendental solution of finite order.*

Further, we will show

Theorem 4. *For any p, q, and n, there is an equation of the form*

(1.1) *any solution of which is transcendental and of order* ∞, *supposed that* $q_0 \geqq 2$.

In Theorems 2–4, we mean by order the one in the sense of Nevanlinna. Now, suppose that n and $R(w)$ be given, and put

$$E=\{(\alpha_1, \cdots, \alpha_n); \text{ equation (1.1) has a rational solution or a solution of finite order}\}.$$

Then we conjecture that the set E would be very small, e.g., it would be of the first Baire category in C^n, supposed that $q_0 \geqq 2$.

2. Proof of Theorem 1. Put

$$L(w)=(2w+1)/(-w).$$

Then the equation

$$y(x+1)=L(y(x))$$

possesses a rational solution

$$y(x)=(x-1)/(-x+2). \tag{2.1}$$

Obviously, the k-th iteration $L^k(w)$ of $L(w)$ is written as

$$L^k(w)=[(k+1)w+k]/[-kw+(1-k)], \qquad k=1,2,\cdots.$$

Choose $\alpha_1, \cdots, \alpha_{q-1}, \alpha_n$ such that $\alpha_1 \cdots \alpha_{q-1}\alpha_n \neq 0$ and, if we write

$$\alpha_n L^n(w)+\alpha_{q-1}L^{q-1}(w)+\cdots+\alpha_1 L(w)=P(w)/Q(w),$$

then $P(w)$ and $Q(w)$ are mutually prime, and further that $\deg[P]=p$, $\deg[Q]=q$. Such choice is obviously possible. Then $y(x)$ in (2.1) is also a solution of the equation

$$\alpha_n y(x+n)+\alpha_{q-1}y(x+q-1)+\cdots+\alpha_1 y(x+1)=P(y(x))/Q(y(x)),$$

which is an equation of the type desired.

3. Proof of Theorem 2. Let ρ be a primitive n-th root of 1. Put

$$L(w)=\rho w/(w+1).$$

Then, the k-th iteration $L^k(w)$ of $L(w)$ is written as

$$L^k(w)=\rho^k w/\{[(\rho^k-1)/(\rho-1)]w+1\} \qquad \text{if } k<n,$$
$$L^n(w)=w.$$

Of course, $q<n$. Choose $\alpha_1, \cdots, \alpha_q$ such that $\alpha_1 \cdots \alpha_q \neq 0$ and, if

$$\alpha_q L^q(w)+\cdots+\alpha_1 L(w)=P_1(w)/Q(w),$$

then $P_1(w)$ and $Q(w)$ are mutually prime polynomials of degree q. Such a choice is possible, obviously. Let $y(x)$ be a solution of the equation

$$y(x+1)=L(y(x)). \tag{3.1}$$

$y(x)$ can be taken as a function of order 1. Then $y(x)$ is also a solution of the equation

$$\begin{cases} \alpha_n y(x+n)+\alpha_q y(x+q)+\cdots+\alpha_1 y(x+1) \\ \qquad =\alpha_n y(x)+P_1(y(x))/Q(y(x))=P(y(x))/Q(y(x)), \end{cases} \tag{3.2}$$

which is an equation to be required, i.e.,

$$\deg[P]=\deg[Q]+1=q+1.$$

4. Proof of Theorem 3. Let σ be a primitive $(n+1)$-th root of 1. Put

$$L(w)=\sigma w/(w+1).$$

Then

$$L^k(w)=\sigma^k w\Big/\left[\frac{\sigma^k-1}{\sigma-1}w+1\right], \qquad k=1, \cdots, n.$$

Choose $\alpha_n, \alpha_{q-1}, \cdots, \alpha_1$ such that $\alpha_n\alpha_{q-1}\cdots\alpha_1\neq 0$ and, if we write

$$\alpha_n L^n(w)+\alpha_{q-1}L^{q-1}(w)+\cdots+\alpha_1 L(w)=P(w)/Q(w),$$

then $P(w)$ and $Q(w)$ are mutually prime, and further that $\deg[P]=p$, $\deg[Q]=q$. Such a choice is obviously possible, and we obtain an equation desired, as in §§2 and 3.

5. Proof of Theorem 4. Consider the equation

$$y(x+n)=R(y(x)). \tag{5.1}$$

Put

$$y(nt)=z(t).$$

Then

$$z(t+1)=y(nt+n)=R(y(nt))=R(z(t)), \tag{5.2}$$

and $z(t)$ is of order ∞ by [2, p. 311, Theorem 1]. Thus $y(x)$ is also of order ∞.

6. A final remark. We conjecture that the equation (1.1) possesses a rational solution or a transcendental solution of finite order if and only if it shares a solution with an equation of the form

$$y(x+1)=[ay(x)+b]/[cy(x)+d],$$

where a, b, c, d are consts, $ad-bc\neq 0$.

References

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