77. *Semigroups and Boundary Value Problems. II*

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1. Introduction. The purpose of this note is to extend our earlier result [5] on the existence of Feller semigroups to a broader class of degenerate elliptic operators.

Let D be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary ∂D and let $C(\bar{D})$ be the space of real-valued continuous functions on $\bar{D}=D\cup\partial D$. A strongly continuous semigroup $\{T_t\}_{t\geqq 0}$ of bounded linear operators on $C(\bar{D})$ is called a *Feller semigroup* on $\bar{D}$ if $\{T_t\}$ satisfies:

$$f\in C(\bar{D}),\quad 0\leqq f\leqq 1 \quad \text{on} \quad \bar{D}\Longrightarrow 0\leqq T_t f\leqq 1 \quad \text{on} \quad \bar{D}.$$

It is known that there corresponds to a Feller semigroup $\{T_t\}_{t\geqq 0}$ on $\bar{D}$ a strong Markov process $\mathcal{X}$ on $\bar{D}$ and that if the paths of $\mathcal{X}$ are continuous, then the infinitesimal generator $\mathfrak{A}$ of $\{T_t\}$ is described analytically as follows (cf. [1], [6]):

i) Let x be a fixed point of the *interior* D of $\bar{D}$. For a C^2-function u in the domain $\mathcal{D}(\mathfrak{A})$ of $\mathfrak{A}$, we have

$$(1)\qquad \mathfrak{A}u(x)=Au(x)$$
$$\equiv \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}(x)+\sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i}(x)+c(x)u(x)$$

where $(a^{ij}(x))\geqq 0$ and $c(x)\leqq 0$.

ii) Let x' be a fixed (regular) point of the *boundary* ∂D of $\bar{D}$ and choose a local coordinate $x=(x_1, x_2, \cdots, x_{N-1}, x_N)$ as $x\in D$ if $x_N>0$ and $x\in\partial D$ if $x_N=0$. For $u\in\mathcal{D}(\mathfrak{A})\cap C^2(\bar{D})$, we have

$$(2)\qquad Lu(x')\equiv \sum_{i,j=1}^{N-1} \alpha^{ij}(x')\frac{\partial^2 u}{\partial x_i\partial x_j}(x')+\sum_{i=1}^{N-1} \beta^i(x')\frac{\partial u}{\partial x_i}(x')$$
$$+\gamma(x')u(x')+\mu(x')\frac{\partial u}{\partial n}(x')-\delta(x')Au(x')$$
$$=0$$

where $(\alpha^{ij}(x'))\geqq 0$, $\gamma(x')\leqq 0$, $\mu(x')\geqq 0$, $\delta(x')\geqq 0$ and $n=(n_1, n_2, \cdots n_N)$ is the unit interior normal to ∂D at x'. The condition L is called a Ventcel's boundary condition.

In this note we consider the following

Problem. *Conversely, given analytic data* (A, L), *can we construct a Feller semigroup* $\{T_t\}_{t\geqq 0}$ *on* $\bar{D}$?

In [5], the author proved that, under the ellipticity condition on A, if a Markovian particle with generator $L^0=\sum_{i,j=1}^{N-1}\alpha^{ij}\,\partial^2/\partial x_i\partial x_j$ goes through the set $M=\{x'\in\partial D\,;\,\mu(x')=0\}$, where no reflection phenomenon occurs, in finite time, then there exists a Feller semigroup $\{T_t\}_{t\geqq 0}$ on $\bar{D}$ whose infinitesimal generator $\mathfrak{A}$ coincides with the minimal closed extension in $C(\bar{D})$ of the restriction of A to the space $\{u \in C^2(\bar{D})\,;\ Lu=0 \text{ on } \partial D\}$.

The purpose of this note is to generalize this result to the case when the operator A is *non-elliptic.*

2. Statement of result. For the differential operator A given by (1), assume that there exists an open subset G of $\boldsymbol{R}^N$, containing $\bar{D}$, such that the coefficients of A satisfy:

$$(3)\quad \begin{cases} 1^\circ & a^{ij} \in C^\infty(G) \quad \text{with } a^{ij}=a^{ji} \text{ and} \\ & \sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geqq 0, \quad x \in G, \quad \xi=(\xi_1, \xi_2, \cdots, \xi_N) \in \boldsymbol{R}^N. \\ 2^\circ & b^i \in C^\infty(G). \\ 3^\circ & c \in C^\infty(G) \quad \text{with } c(x) \leqq 0 \text{ in } D. \end{cases}$$

Setting

$$\rho(x) = \operatorname{dist}(x, \partial D) \quad (x \in \bar{D}),$$
$$b(x) = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x)\frac{\partial \rho}{\partial x_i}(x),$$

we divide the boundary ∂D into four disjoint subsets (cf. [3]):

$$\Sigma_3 = \left\{x' \in \partial D\,;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j > 0\right\},$$
$$\Sigma_2 = \left\{x' \in \partial D\,;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j = 0,\ b(x') < 0\right\},$$
$$\Sigma_1 = \left\{x' \in \partial D\,;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j = 0,\ b(x') > 0\right\},$$
$$\Sigma_0 = \left\{x' \in \partial D\,;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j = 0,\ b(x') = 0\right\}.$$

The fundamental hypothesis concerning A is the following

(H) *each Σ_i $(i=1, 2, 3)$ consists of a finite number of connected hypersurfaces.*

Note that $\Sigma_2 \cup \Sigma_3$ coincides with the set of all regular points (cf. [4]).

For the Ventcel's boundary condition L given by (2), assume that the coefficients of L satisfy:

$$(4)\quad \begin{cases} 1^\circ & \alpha^{ij} \text{ are the components of a } C^\infty \text{ symmetric contravariant} \\ & \text{tensor of type } (2, 0) \text{ on } \Sigma_2 \cup \Sigma_3 \text{ and} \\ & \sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j \geqq 0, \qquad x' \in \Sigma_2 \cup \Sigma_3,\ \xi' \in T^*_{x'}(\Sigma_2 \cup \Sigma_3). \\ 2^\circ & \beta^i \in C^\infty(\Sigma_2 \cup \Sigma_3). \\ 3^\circ & \gamma \in C^\infty(\Sigma_2 \cup \Sigma_3) \quad \text{with } \gamma(x') \leqq 0 \text{ on } \Sigma_2 \cup \Sigma_3. \\ 4^\circ & \mu \in C^\infty(\Sigma_2 \cup \Sigma_3) \quad \text{with } \mu(x') \geqq 0 \text{ on } \Sigma_2 \cup \Sigma_3. \\ 5^\circ & \delta \in C^\infty(\Sigma_2 \cup \Sigma_3) \quad \text{with } \delta(x') \geqq 0 \text{ on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

To state hypotheses concerning L, we introduce some notation and definitions.

Following [2], we say that a tangent vector $X=\sum_{j=1}^{N-1}\gamma^j(\partial/\partial x_j)$ at $x'\in\Sigma_3$ is *subunit* for $L^0=\sum_{i,j=1}^{N-1}\alpha^{ij}(\partial^2/\partial x_i\partial x_j)$ if

$$\left(\sum_{j=1}^{N-1}\gamma^j\eta_j\right)^2\leqq\sum_{i,j=1}^{N-1}\alpha^{ij}(x')\eta_i\eta_j,\qquad \eta=\sum_{j=1}^{N-1}\eta_j dx_j\in T^*_{x'}(\Sigma_3).$$

For $x'\in\Sigma_3$ and $\rho>0$, we define a "non-Euclidean ball" (of radius ρ about x') $B_{L^0}(x',\rho)$ to be the set of all points $y'\in\Sigma_3$ which can be joined to x' by a Lipschitz path $\gamma:[0,\rho]\to\Sigma_3$ for which $(d/dt)\gamma(t)$ is a subunit vector for L^0 at $\gamma(t)$ for almost every t. We denote by $B_E(x',\rho)$ an ordinary Euclidean ball of radius ρ about x'.

The hypothesis concerning L on Σ_3 is the following

(A.1) *The operator A is elliptic near Σ_3 and there exist constants $0<\varepsilon_1\leqq 1$ and $C_1>0$ such that for sufficiently small $\rho>0$ we have*

$$B_E(x',\rho)\subset B_{L^0}(x',C_1\rho^{\varepsilon_1}),\qquad x'\in M=\{x'\in\Sigma_3;\ \mu(x')=0\}.$$

The intuitive meaning of hypothesis (A.1) is that a Markovian particle with generator L^0 *goes through* the set M, where no reflection phenomenon occurs, in finite time (cf. [5], Remark 2.5).

In a neighborhood of Σ_2, we can write A uniquely in the form: $A=A_0(\partial^2/\partial n^2)+A_1(\partial/\partial n)+A_2$ where A_j $(j=0,1,2)$ is a differential operator of order j acting along the parallel surfaces of Σ_2. Note that by hypothesis (H) the restriction $A_2|_{\Sigma_2}$ of A_2 to Σ_2 is a second order differential operator with non-positive principal symbol, and that $\mu\geqq 0$ and $b<0$ on Σ_2. Thus, for $x'\in\Sigma_2$ and $\rho>0$, we can define a "non-Euclidean ball" $B_{L^0-(\mu/b)(A_2|_{\Sigma_2})}(x',\rho)$ in the same manner as $B_{L^0}(x',\rho)$, replacing Σ_3 and L^0 by Σ_2 and $L^0-(\mu/b)(A_2|_{\Sigma_2})$ respectively.

The hypothesis concerning L on Σ_2 is the following

(A.2) *There exist constants $0<\varepsilon_2\leqq 1$ and $C_2>0$ such that for sufficiently small $\rho>0$ we have*

$$B_E(x',\rho)\subset B_{L^0-(\mu/b)(A_2|_{\Sigma_2})}(x',C_2\rho^{\varepsilon_2}),\qquad x'\in\Sigma_2.$$

The intuitive meaning of hypothesis (A.2) is that a Markovian particle with generator $L^0-(\mu/b)(A_2|_{\Sigma_2})$ diffuses *everywhere* in Σ_2 in finite time.

The boundary condition L is said to be *transversal* on $\Sigma_2\cup\Sigma_3$ if

$$\mu(x')+\delta(x')>0\qquad\text{on }\Sigma_2\cup\Sigma_3.$$

Now we can state the main result, which is a generalization of Theorem 3 of [5].

Theorem. *Let the differential operator A satisfy* (3) *and hypothesis* (H) *and let the boundary condition L satisfy* (4) *and be transversal on $\Sigma_2\cup\Sigma_3$. Suppose that hypotheses* (A.1), (A.2) *are satisfied. Then there exists a Feller semigroup $\{T_t\}_{t\geqq 0}$ on $\bar{D}$ whose infinitesimal generator $\mathfrak{A}$ coincides with the minimal closed extension in $C(\bar{D})$ of the restriction of A to the space $\{u\in C^\infty(\bar{D});\ Lu=0$ on $\Sigma_2\cup\Sigma_3\}$.*

3. Idea of proof. Hypotheses (A.1), (A.2) imply that there exists a strong Markov process $\mathcal{Y}$ on $\Sigma_2\cup\Sigma_3$ and the transversality of L implies that $\mathcal{Y}$ is the "trace" on $\Sigma_2\cup\Sigma_3$ of trajectories of a strong

Markov process on $\bar{D}$. On the other hand, the intuitive meaning of hypothesis (H) is that a Markovian particle with generator A (A-diffusion) does not diffuse in D all the time, but it either dies or attains the set $\Sigma_2 \cup \Sigma_3$ some time or other. Therefore we can "piece out" $\mathcal{Y}$ with A-diffusion in D to construct a strong Markov process $\mathcal{X}$ on $\bar{D}$ and hence a Feller semigroup $\{T_t\}_{t \geqq 0}$ on $\bar{D}$.

The details will be published elsewhere.

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